

# LOSS OF BOUNDARY CONDITIONS FOR FULLY NONLINEAR PARABOLIC EQUATIONS WITH SUPERQUADRATIC GRADIENT TERMS

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## Abstract

We study whether the solutions of a fully nonlinear, uniformly parabolic equation with superquadratic growth in the gradient satisfy initial and homogeneous boundary conditions in the classical sense, a problem we refer to as the classical Dirichlet problem. Our main results are: the nonexistence of global-in-time solutions of this problem, depending on a specific largeness condition on the initial data, and the existence of local-in-time solutions for initial data  $C^1$  up to the boundary. Global existence is known when boundary conditions are understood in the viscosity sense, what is known as the generalized Dirichlet problem. Therefore, our result implies loss of boundary conditions in finite time. Specifically, that a solution satisfying homogeneous boundary conditions in the viscosity sense eventually becomes strictly positive at some point of the boundary.

**Keywords:** Loss of boundary conditions, generalized Dirichlet problem, viscosity solutions, gradient blow-up, fully nonlinear parabolic equations, viscous Hamilton-Jacobi equations, nonlinear eigenvalues.

**MSC (2010):** 35D40, 35K55, 35K20, 35P30.

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## 1 Introduction and main results

The present article is a contribution to the study of qualitative properties of viscosity solutions of the so-called Cauchy-Dirichlet problem for the following fully nonlinear parabolic equation with superquadratic growth in the term with gradient dependence:

$$u_t - \mathcal{M}^-(D^2u) = |Du|^p \quad \text{in } \Omega \times (0, T), \quad (1.1)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (1.2)$$

$$u(x, 0) = u_0(x) \quad \text{in } \overline{\Omega}. \quad (1.3)$$

We assume  $\Omega \subset \mathbb{R}^n$  is a bounded domain satisfying uniform interior and exterior sphere conditions and  $p > 2$  (except for certain remarks regarding the case  $p \leq 2$  made in this introduction). Here  $\mathcal{M}^-$  denotes one of Pucci's extremal operators, which are defined as follows: let  $A, X \in S(n)$ , the symmetric  $n \times n$  matrices equipped with the usual ordering,  $I$  denote the identity matrix, and  $0 < \lambda < \Lambda$ . Then

$$\begin{aligned} \mathcal{M}^-(X) &= \mathcal{M}^-(X, \lambda, \Lambda) = \inf\{\text{tr}(AX) \mid \lambda I \leq A \leq \Lambda I\}, \\ \mathcal{M}^+(X) &= \mathcal{M}^+(X, \lambda, \Lambda) = \sup\{\text{tr}(AX) \mid \lambda I \leq A \leq \Lambda I\}. \end{aligned}$$

Alternatively, if we denote by  $\lambda_i = \lambda_i(X)$  the eigenvalues of  $X$ , then

$$\begin{aligned} \mathcal{M}^-(X) &= \lambda \sum_{\lambda_i > 0} \lambda_i + \Lambda \sum_{\lambda_i < 0} \lambda_i \\ \mathcal{M}^+(X) &= \Lambda \sum_{\lambda_i > 0} \lambda_i + \lambda \sum_{\lambda_i < 0} \lambda_i. \end{aligned}$$

The Dirichlet condition (1.2) will be considered both in the classical sense and in the generalized sense of viscosity solutions. We will stress the distinction when necessary. Precise definitions and more on this later in this introduction. On the other hand, condition (1.3) is meant in the classical (pointwise) sense. See Remark 2.5.

We assume the compatibility condition

$$u_0(x) = 0 \quad \text{for all } x \in \partial\Omega$$

is satisfied in the classical sense, as well as  $u_0 \in C^1(\overline{\Omega})$ . This regularity is not strictly necessary for all of our results, but helps to establish a connection between existence and nonexistence of local and global-in-time solutions, respectively. See Remark 6.9. Also, we assume  $u_0 \geq 0$  without loss of generality, since (1.1) is invariant with respect to additive constants.

Equation (1.1) can be seen as a generalization of the so-called viscous Hamilton-Jacobi equation,

$$u_t - \Delta u = |Du|^p \text{ in } \Omega \times (0, T). \quad (1.4)$$

For  $p = 2$ , this corresponds to the deterministic Kardar-Parisi-Zhang equation, proposed by these authors in [20] as a model for the profile of a growing interface. Mathematically, it is of interest because it is the simplest model of a parabolic equation with nonlinear dependence on the gradient, as well as a viscosity approximation of a first-order Hamilton-Jacobi equation (see [16], Ch. 10).

For equation (1.4), including all values  $p > 0$ , it is well-known that there exists a unique, maximal-in-time classical solution  $u \in C^{1,\alpha}(\Omega \times [0, T^*])$  for some  $\alpha > 0$  and  $0 < T^* \leq \infty$ , assuming sufficient regularity for  $\Omega$  and for the initial and boundary data ([18], Ch. 7).

In [29] the nonexistence of global, classical solutions of problem (1.4)-(1.2)-(1.3) is proved when  $p > 2$  and  $u_0 \in C^1(\bar{\Omega})$  and is suitably large. It is also shown here that this implies the occurrence of *gradient blow-up* (GBU, for short). GBU is said to occur in finite time  $0 < T < \infty$  if a solution  $u$  satisfies

$$\sup_{[0,T] \times \Omega} u < \infty, \quad \lim_{t \rightarrow T} \sup_{x \in \Omega} |Du(x, t)| = \infty.$$

A version of equation (1.4) containing a more general gradient term with superquadratic growth is studied in [1] in the context of weak solutions, for irregular initial data. The notable result is the nonexistence of global-in-time weak solutions with initial data  $u_0$  a positive, bounded measure and suitably large.

In both [1] and [29], the largeness condition on  $u_0$  is (roughly speaking) given in terms of an  $L^2$ -product of  $u_0$  with the principal eigenfunction of the Laplacian. The condition that appears in our Theorem 1.2 is essentially the same as the one in [29]. An alternative proof of global nonexistence for (1.4) given in [27], Theorem 40.2, uses a weaker condition on  $u_0$ . In this proof it is enough to consider the  $L^q$ -norm for any  $q \geq 1$ , but the argument does not adapt to more general nonlinearities.

There are different extensions of the results of [29]. Still in the context of classical solutions, the existence of global solutions and their large-time (or asymptotic) behavior for equation (1.4) with nontrivial right-hand side is studied in [30]. For equation

$$u_t - \Delta u = |Du|^p + \lambda h(x) \quad \text{in } \Omega \times (0, T), \quad (1.5)$$

where  $\lambda \geq 0$ ,  $h \in C^1(\bar{\Omega})$ ,  $h \geq 0$ , a complete description of the asymptotic behavior is given when  $u_0, h$  are radially symmetric and  $\Omega = B_R(0)$ , for some  $R > 0$ : in this case, for  $h \not\equiv 0$ , there exists a  $\lambda^* > 0$  such that

- if  $0 \leq \lambda < \lambda^*$ , then (1.5) has a global solution which converges to the solution of the steady-state equation

$$-\Delta v = |Dv|^p + \lambda h(x) \quad \text{in } \Omega, \quad (1.6)$$

which additionally satisfies  $v \in C^1(\overline{\Omega})$ .

- if  $\lambda = \lambda^*$ , then  $u$  converges to a solution  $v \notin C^1(\overline{\Omega})$  for any  $u_0 \in C^1(\overline{\Omega})$  with  $u_0 \leq v$ . This implies GBU in *infinite time*, i.e.,

$$\limsup_{t \rightarrow \infty} \|Du(\cdot, t)\|_{\infty} = \infty.$$

- if  $\lambda > \lambda^*$ , then (1.6) has no solution and GBU in finite time occurs for *any*  $u_0 \in C^1(\overline{\Omega})$ .

In the case of a general, bounded domain  $\Omega \subset \mathbb{R}^n$  only a partial description is available.

Some of these results have been extended to equations with degenerate diffusion (i.e., with  $\Delta_p$  in place of  $\Delta$ ) in [4] in the context of weak solutions. Other questions, such as determining precise blow-up rates, profiles and sets are addressed in [33], [4], [24]. See also [27], Ch. IV, and the references therein.

Equation (1.4) has also been studied from the viewpoint of viscosity solutions, in which a generalized notion of boundary conditions exists. The relevant phenomenon in this context is known as *loss of boundary conditions* (LOBC, for short). More precisely, (1.2) is said to hold in the viscosity sense for (1.1) if

$$\min(u_t - \mathcal{M}^-(D^2u) - |Du|^p, u) \leq 0, \quad \text{and} \quad (1.7)$$

$$\max(u_t - \mathcal{M}^-(D^2u) - |Du|^p, u) \geq 0, \quad (1.8)$$

while loss of boundary conditions are said to occur whenever (1.2) is not satisfied in the classical sense (see [12]).

In [7] it is proved that the Cauchy-Dirichlet problem for a class of fully-nonlinear equations which includes (1.1) admits a globally defined, continuous viscosity solution, assuming boundary conditions are understood in the viscosity sense. The result follows from a strong comparison principle (SCR, for short) proved in the same work and Perron's method. The relevance of this result is in the case where we have superquadratic gradient growth. It is shown in this same work that in the subquadratic case there is no LOBC, hence the classical comparison result of [12] applies. A one dimensional example of LOBC is also provided which, in contrast to those furnished by our result, satisfies time-dependent boundary data. As these results apply directly to the problem under our consideration, we review some of them in Section 2 for convenience.

Building on the existence of global solutions of the generalized Dirichlet problem, a natural question is to determine their large-time behavior. In this direction again there is an important distinction between the sub- and superquadratic cases, which are studied rather thoroughly in [8] and [32], respectively. Consider equation

$$u_t - \Delta u + |Du|^p = f(x) \quad \text{in } \Omega \times (0, T), \quad (1.9)$$

where

$$u(x, t) = \varphi(x) \quad \text{on } \partial\Omega \times (0, T),$$

is satisfied *in the viscosity sense*,  $f \in C(\Omega)$ ,  $\varphi \in C(\partial\Omega)$ , and  $\varphi(x) = u_0(x)$  for all  $x \in \partial\Omega$ . In the superquadratic case,  $p > 2$ , there are two possibilities: if the corresponding steady-state equation

$$-\Delta v + |Dv|^p = f(x) \quad \text{in } \Omega \quad (1.10)$$

has a bounded subsolution, then there exists a solution  $u_\infty$  of (1.10) and  $u(x, t) \rightarrow u_\infty$  on  $\bar{\Omega}$ . If (1.10) fails to have bounded subsolutions, one must introduce the so-called *ergodic problem* with state-constraint boundary conditions:

$$-\Delta v + |Dv|^p = f(x) + c \quad \text{in } \Omega, \quad (1.11)$$

$$-\Delta v + |Dv|^p \geq f(x) + c \quad \text{in } \partial\Omega. \quad (1.12)$$

Here  $c \in \mathbb{R}$  is the so-called *ergodic constant*, and is an unknown in problem (1.11) together with  $v$ . Existence and uniqueness of solutions  $(c, v)$  of (1.11) are studied in [21]:  $c$  is unique while  $v$  is unique up to an additive constant. Convergence of  $u(x, t) + ct$  to  $v$  where  $(c, v)$  is a solution of (1.11), as well as LOBC is then analyzed.

The behavior in the subquadratic case is more complicated. It depends also on whether  $1 < p \leq 3/2$  or  $3/2 < p \leq 2$  and becomes necessary to introduce the following problem, also studied in [21], as an analogue of (1.10) and (1.11):

$$-\Delta v + |Dv|^p = f(x) + c \quad \text{in } \Omega, \quad (1.13)$$

$$v(x) \rightarrow \infty \quad \text{as } x \rightarrow \partial\Omega. \quad (1.14)$$

We refer the reader to [8].

A different type of result concerning large-time behavior is given in [25]. It is shown that there exist constants  $K, \lambda$  and  $C$  such that the solution of the generalized Dirichlet problem for (1.9) with homogeneous boundary data and *any* compatible initial data  $u_0 \in C(\bar{\Omega})$  satisfies, for every  $t \geq K\|u_0\|_\infty$ ,

$$u(\cdot, t) \in W^{1,\infty}(\Omega), \quad \text{and} \quad \|u(\cdot, t)\|_\infty + \|Du(\cdot, t)\|_\infty \leq Ce^{-\lambda t}.$$

In particular, after some finite time, the solution  $u$  satisfies the boundary data in the classical sense. This property is then applied to the interesting problem of the null controllability of (1.9).

Regarding regularity of solutions, it is proved in [10] that if  $u$  is a bounded, upper-semicontinuous viscosity subsolution of the (possibly degenerate) elliptic equation

$$-\text{tr}(A(x)D^2u) + \lambda u + |Du|^p = f(x) \quad \text{for all } x \in \Omega, \quad (1.15)$$

where  $p > 2$ ,  $\Omega \subset \mathbb{R}^n$  is a regular domain,  $\lambda > 0$ , and  $A : \Omega \mapsto S(n)$  and  $f$  satisfy fairly standard assumptions, then  $u$  is *globally* Hölder continuous with exponent  $\alpha = p-2/p-1$  (i.e.,  $u \in C^{0,p-2/p-1}(\bar{\Omega})$ ). As noted in [6], the result above is surprising, since most regularity results apply to actual *solutions* of *uniformly* elliptic equations that satisfy *subquadratic* growth conditions, none

of which points are met in the assumed hypotheses. The authors of [10] go on to prove interior Lipschitz bounds for *solutions* of (1.15) by the so-called *weak-Bernstein* method introduced in [5]. These results are valid for fully-nonlinear equations satisfying hypotheses which are discussed in detail in [6].

In [6] a slight simplification of the proof of Hölder regularity is provided, and the relation to the solvability of Dirichlet problem is analyzed. In short, if a general boundary condition  $u = \varphi$  with  $\varphi \in C(\partial\Omega)$  is assumed in the viscosity sense, an additional reason for the occurrence of LOBC is that  $\varphi$  might not have the regularity of  $u$ . This is, of course, irrelevant to the case of homogeneous boundary data.

Time-dependent versions of these regularity results are proven in [3], though they require the additional assumption that

$$u_t \geq -C \quad \text{for all } (x, t) \in \Omega \times (0, T)$$

for some  $C \geq 0$  be satisfied *in the viscosity sense*. This means that: for all  $(x, t) \in \Omega \times (0, T)$ , if  $(a, \xi, X) \in \mathcal{P}^{2,+}u(x, t)$ , the parabolic superjet of a subsolution  $u$  (see, e.g., [12] for definitions), then  $a \geq -C$ .

#### Main results

Our main results are the following. We begin by proving the existence of solutions of (1.1)-(1.2)-(1.3) that for a small time satisfy the boundary data in the classical sense. The existence time depends only on a gradient bound for the initial data, the remaining constants usually considered universal.

**Theorem 1.1.** *Let  $u_0 \in C^1(\overline{\Omega})$ . There exists a  $T^* > 0$ , depending only on  $\Lambda, \lambda, n, \Omega$  and  $\|u_0\|_{C^1(\overline{\Omega})}$ , such that the viscosity solution of (1.1) in  $\Omega \times (0, T^*)$  satisfies (1.2) and (1.3) in the classical sense.*

Since we already have the existence result of [7], we need only show that (1.2) is satisfied in the classical sense. For this we use a barrier argument, following the construction of comparison functions used in [4] to show local existence under a slightly different strategy.

Next we prove the nonexistence of global solutions to the classical Dirichlet problem when  $\Omega = B_1(0)$  and the initial data is radially symmetric, and suitably large. Again, thanks to the global existence result of [7], this implies the occurrence of LOBC.

**Theorem 1.2.** *Let  $u_0 \in C^1(\overline{B_1(0)})$  be a radial function. Then, there exist positive constants  $\delta = \delta(\lambda, \Lambda, n)$  and  $M = M(\lambda, \Lambda, n, p)$  such that, if*

$$\int_{\delta}^{1-\delta} u_0(r) - \frac{1}{2} \|u_0\|_{\infty} dr > M \quad (1.16)$$

*then the solution  $u$  of (1.1)-(1.2)-(1.3) with  $\Omega = B_1(0)$  and initial data  $u_0$  has LOBC at some finite time  $T = T(u_0)$ .*

The proof of Theorem 1.2 uses key ideas from that of Theorem 2.1 in [29]. The main difficulty in adapting this proof is its crucial use of the divergence structure of the Laplacian by repeatedly using integration by parts. We remedy this problem by using the divergence form of the Pucci operator, available for radial solutions (see, e.g., [17]), and the regularization by inf-sup-convolution introduced in [22]. Combining these techniques we obtain an equation in divergence form which is satisfied point-wise and all of whose terms are integrable. Afterwards, the main complications are keeping track of the terms which depend on the regularization parameters and providing estimates which are independent of these. We also adapt a weighted, one-dimensional version of Poincaré's inequality, and make use of different results from [9] and [14] regarding the principal eigenvalue problem for the Pucci operator.

This result is extended to show LOBC occurs for solutions of (1.1) in a sufficiently regular bounded domain in Corollary 6.10, and then to equations with more general nonlinearities, first in the radially symmetric case, then also for a bounded domain as above. The most general result is the following. Consider

$$u_t - F(D^2u) = f(Du) \text{ in } \Omega \times (0, T), \quad (1.17)$$

where  $F : S(n) \rightarrow \mathbb{R}$  is uniformly elliptic, i.e.,

$$\mathcal{M}^-(X - Y) \leq F(X) - F(Y) \leq \mathcal{M}^+(X - Y) \quad \text{for all } X, Y \in S(n), \quad (1.18)$$

and vanishes at zero, i.e.,  $F(0) = 0$ , and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies  $f(\xi) \geq |\xi|^2 h(|\xi|)$  for all  $\xi \in \mathbb{R}^n$ , where  $h : \mathbb{R} \rightarrow \mathbb{R}$  is positive, nondecreasing, convex, and grows more slowly than any positive power. Precise hypotheses on  $h$  are given in Section 7.

**Theorem 1.3.** *Let  $\Omega$  be a bounded domain satisfying a uniform interior sphere condition, and assume that  $F$ ,  $f$ , and  $h$  are as described above. If additionally  $h$  satisfies*

$$\int_1^\infty \frac{1}{sh(s)} ds < \infty, \quad (1.19)$$

*then there exists  $u_0 \in C^1(\overline{\Omega})$ , with  $u_0 \geq 0$  and  $u_0|_{\partial\Omega} = 0$ , such that LOBC occurs for solutions of (1.17)-(1.2)-(1.3) in some finite time  $T = T(u_0)$ .*

This result follows more or less easily from Theorem 1.2 and the main ideas used in its proof, as do the other extensions given in the final section.

The organization of the article is as follows. In Section 2 we briefly review the results of [7] which are directly used in our work. Section 3 is devoted to the proof of Theorem 1.1. In Sections 4 and 5 we develop some preliminaries regarding regularization by inf-sup-convolution and the radial form of the extremal operator, respectively, which lead us to the approximate equation we use to prove the nonexistence result. Section 6 contains the proof of our main result in the radially symmetric case, Theorem 1.2, and its extension to a bounded domain. This is the core of our work. Finally, in Section 7 we provide extensions to more general equations, including the proof of Theorem 1.3.

## 2 Comparison, existence and uniqueness

Existence and uniqueness for the so-called generalized Dirichlet problem for

$$u_t + G(x, t, Du, D^2u) = 0 \quad \text{in } \Omega \times (0, T), \quad (2.1)$$

where the boundary condition

$$u = g \quad \text{on } \partial\Omega \times (0, T) \quad (2.2)$$

is understood in the viscosity sense, is proven in Theorem 5.1 in [7]. For convenience, In this section we quote the main results of this work, as well as a couple of remarks relevant to our purposes. Here  $G$  is a continuous function that satisfies the degenerate ellipticity condition,

$$G(x, t, \xi, X) \leq G(x, t, \xi, Y) \quad \text{if } X \geq Y, \quad (2.3)$$

for all  $x \in \bar{\Omega}$ ,  $t \in [0, T]$ ,  $\xi \in \mathbb{R}^n$  and  $X, Y \in S(n)$ , together with two key hypothesis, for which we must introduce additional notation. Note that condition (2.3) uses the opposite sign convention than the one used in (1.18). See also the discussion at the beginning of Subsection 7.1.

Let  $h_1 : [0, \infty) \rightarrow [0, \infty)$  be a continuous function. We say  $h_1$  satisfies property (P) if the following hold:

$$(i) \quad \int_0^\infty \frac{s}{h_1(s)} ds < \infty,$$

$$(ii) \quad \text{for any } C > 0, s \text{ large enough and } L \geq 1, \text{ the map } L \mapsto h_1(Ls) - CL^2h_1(s) \text{ is increasing,}$$

$$(iii) \quad \text{for any } C, \tilde{C} > 0, \text{ there exists } \bar{s} > 0, \bar{L} \geq 1 \text{ such that}$$

$$h_1(Ls) - CL^2h_1(s) \geq \tilde{C}Ls \quad \text{for } s \geq \bar{s}, L \geq \bar{L}. \quad (2.4)$$

The key assumptions on  $G$  as the following:

(H1) There exists constants  $C_1, C_2 > 0$  and a continuous function  $h_1$  satisfying property (P) such that, for all  $x \in \bar{\Omega}$ ,  $t \in [0, T]$ ,  $\xi \in \mathbb{R}^n$  and  $X \in S(n)$ , we have

$$G(x, t, \xi, X) \geq -C_1 - C_2\|X\| + h_1(|\xi|). \quad (2.5)$$

(H2) For any  $\epsilon > 0$ , there exists  $0 < \mu_\epsilon < 1$  converging to 1 as  $\epsilon \rightarrow 0$  such that

$$G(y, s, \xi_2, Y) - G(x, t, \mu_\epsilon^{-1}\xi_1, \mu_\epsilon^{-1}X) \leq o(1)$$

for all  $x, y \in \bar{\Omega}$ ,  $t, s \in [0, T]$ ,  $\xi_1, \xi_2 \in \mathbb{R}^n$  and for all  $X, Y \in S(n)$  satisfying the following properties for some  $K > 0$  and a sufficiently small  $\eta > 0$ :

$$-\frac{K\eta}{\epsilon^2}I_{2n} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{o(1)}{\epsilon^2} \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix} + o(1)I_{2n}, \quad (2.6)$$

$$|\xi_1 - \xi_2| \leq K\epsilon \min\{|\xi_1|, |\xi_2|\}. \quad (2.7)$$

$$|x - y| + |t - s| < \epsilon. \quad (2.8)$$



The main result is the following:

**Theorem 2.1** (Strong Comparison Result - SCR). *Assume  $u_0 \in C(\overline{\Omega})$ , and let  $u$  and  $v$  be respectively a bounded upper-semicontinuous supersolution (USC, for short) and a bounded lower-semicontinuous (LSC) supersolution of (2.1)-(2.2)-(1.3), where  $G$  satisfies hypotheses (H1) and (H2). Then  $u \leq v$  in  $\Omega \times [0, T]$ . Moreover, if we define  $\tilde{u}$  on  $\overline{\Omega} \times [0, T]$  by setting*

$$\tilde{u}(x, t) = \begin{cases} \limsup_{\substack{(y,s) \rightarrow (x,t) \\ (y,s) \in \Omega \times (0,T)}} u(y, s) & \text{on } \partial\Omega \times (0, T], \\ u(x, t) & \text{otherwise,} \end{cases} \quad (2.9)$$

*and similarly define  $\tilde{v}$ , then  $\tilde{u}$  and  $\tilde{v}$  are still respectively a bounded USC subsolution and a bounded LSC supersolution of (2.1)-(2.2)-(1.3) and  $\tilde{u} \leq \tilde{v}$  in  $\overline{\Omega} \times [0, T]$ .*

As is standard, existence is proven by combining this result with Perron's method of sub- and supersolutions.

*Remark 2.2.* When Theorem 2.1 is used to compare *continuous* sub- and supersolutions, comparison holds up to the boundary without having to redefine the functions as in (2.9).

*Remark 2.3.* The lower bound of (2.5) implies that the gradient nonlinearity has the opposite sign to that of (1.1). However, the results proved for

$$u_t - \mathcal{M}^+(D^2u) + |Du|^p = 0 \quad \text{in } \partial\Omega \times (0, T)$$

are valid for (1.1) provided we exchange the role of sub- and supersolutions. Indeed,  $u$  is a subsolution of the above equation if and only if  $-u$  is a supersolution of (1.1). This is already noted in Remark 3.2 of [7]. We note also that in [7] there is no requirement that the solution be nonnegative, as there is in the proofs of gradient blow-up given in [29].

We will verify that hypotheses that (2.5) and (2) apply to the equations considered in this work (after the appropriate sign change) in Section 7.

*Remark 2.4.* Following the exchange of sub- and supersolutions mentioned in the previous remark, it follows from Proposition 3.1 in [7] that any supersolution  $v$  of (1.1) satisfies  $v \geq 0$  on  $\partial\Omega \times (0, T)$  in the classical sense for any given  $T > 0$ . Hence, if LOBC occurs, as we prove later, then the solution satisfying (1.2) in the generalized sense must become strictly positive at some point of the boundary.

*Remark 2.5.* As mentioned in the introduction, the initial condition (1.3) is always meant in the classical sense. There is no loss of generality in this assumption. It is a consequence of [23], Lemma 4.1, that there is no LOBC on the *bottom* of the parabolic domain,  $\overline{\Omega} \times \{t = 0\}$ .

*Remark 2.6.* An easy but important consequence of the comparison result is that solutions  $u$  of (1.1)-(1.3) are uniformly bounded and nonnegative. Indeed,

for  $u_0 \geq 0$ ,  $\underline{v} \equiv 0$  and  $\bar{v} \equiv \sup_{\bar{\Omega}} u_0$  are respectively sub- and supersolutions, so by comparison we have

$$0 \leq u(x, t) \leq \sup_{\bar{\Omega}} u_0 \quad \text{for all } x \in \bar{\Omega}, 0 \leq t \leq T. \quad (2.10)$$

In particular,  $\|u\|_{\infty} \leq \|u_0\|_{\infty}$ .

### 3 Existence of local solutions

We follow the construction of the comparison functions used to prove local existence of solutions for a related problem in [4], accounting for the presence of the extremal operators and providing additional detail regarding the choice of constants.

*Proof of Theorem 1.1: Step 1: A time-independent barrier.* We define a time-independent comparison function in a neighborhood of a given, fixed  $x_0 \in \Omega$ , using the exterior sphere condition, and prove that it is a supersolution of (1.1). We will check the initial and boundary conditions in a later step.

From the exterior sphere condition there exists a ball of radius  $\rho > 0$  centered at  $x_1 \notin \bar{\Omega}$ , tangent to  $\partial\Omega$  at  $x_0$ . We will employ the radial variables  $r = |x - x_1|$ , where  $\rho < r < \rho + \eta$  for some  $\eta > 0$ , and  $s = |x - x_1| - \rho$ . In this and the following steps we will compare the solution  $u$  to different functions in the annulus  $\{0 < s < \eta\} \cap \bar{\Omega}$ .

Let  $\varphi(s) = s(s + \mu)^{-\beta}$  with  $\mu, \beta > 0$  to be chosen later, and define

$$\bar{v}(x) = \varphi(|x - x_1| - \rho) = \varphi(s). \quad (3.1)$$

We compute

$$\begin{aligned} \varphi'(s) &= [(1 - \beta)s + \mu](s + \mu)^{-\beta-1} > 0, \\ \varphi''(s) &= -\beta[(1 - \beta)s + 2\mu](s + \mu)^{-\beta-2} < 0. \end{aligned}$$

Hence, the extremal operator takes the form

$$\begin{aligned} \mathcal{M}^-(D^2\bar{v})(s) &= \Lambda\varphi''(s) + \lambda \left( \frac{n-1}{s+\rho} \right) \varphi'(s) \\ &= -\Lambda\beta[(1 - \beta)s + 2\mu](s + \mu)^{-\beta-2} \\ &\quad + \lambda \left( \frac{n-1}{s+\rho} \right) [(1 - \beta)s + \mu](s + \mu)^{-\beta-1}, \end{aligned}$$

and  $\bar{v}$  is a super solution if

$$-\mathcal{M}^-(D^2\bar{v}) \geq |\nabla\bar{v}|^p = |\varphi'|^p.$$

That is, if

$$\begin{aligned} & \left[ \Lambda\beta[(1-\beta)s + 2\mu] - \lambda \left( \frac{n-1}{s+\rho} \right) [(1-\beta)s + \mu](s + \mu) \right] (s + \mu)^{-\beta-2} \\ & \geq [(1-\beta)s + \mu]^p (s + \mu)^{-p(\beta+1)}. \end{aligned}$$

Here we have factored the leading term  $(s + \mu)^{-\beta-2}$  in the left-hand side. We proceed to show that its coefficient  $K$  is positive for the right choices of  $\mu$  and  $\beta$ .

Setting  $\eta = \mu$  and using only that  $0 < \beta < 1$  and  $0 < s < \eta = \mu$ , we have

$$K > 2\Lambda\beta\mu - \lambda \left( \frac{n-1}{s+\rho} \right) ((1-\beta)\mu + \mu)\mu$$

Hence, to have  $K > 0$  it is sufficient that

$$\mu < \frac{\beta}{2-\beta} \left( \frac{2\Lambda\rho}{\lambda(n-1)} \right). \quad (3.2)$$

Next, we verify that

$$K(s + \mu)^{-\beta-2} \geq [(1-\beta)s + \mu]^p (s + \mu)^{-p(\beta+1)}. \quad (3.3)$$

Again  $0 < \beta < 1$  implies

$$[(1-\beta)s + \mu]^p \leq (s + \mu)^p,$$

then

$$\begin{aligned} [(1-\beta)s + \mu]^p (s + \mu)^{-p(\beta+1)} & \leq (s + \mu)^p (s + \mu)^{-p-p\beta} \\ & = (s + \mu)^{-p\beta}. \end{aligned}$$

Hence (3.3) holds if  $(s + \mu)^{-p\beta} \leq K(s + \mu)^{-\beta-2}$ , that is, if

$$K^{-1} \leq (s + \mu)^{\beta(p-1)-2}. \quad (3.4)$$

Setting  $\beta < \frac{1}{2(p-1)}$  gives  $\beta(p-1) - 2 < -\frac{3}{2}$ , so that the term on the right is singular. This precise value of  $\beta$  will be useful in a moment. Using once more that  $0 < s < \mu$ , it is sufficient to have

$$K^{-1} \leq (2\mu)^{\beta(p-1)-2}. \quad (3.5)$$

We recall that  $K$ , the coefficient defined above, also depends on  $\mu$ . However, from the above computations we have that for small  $\mu$ ,

$$K \geq 2\Lambda\beta\mu - \lambda \left( \frac{n-1}{s+\rho} \right) ((1-\beta)\mu + \mu)\mu \geq C_1\mu - C_2\mu^2,$$

hence  $K^{-1} = O(\mu^{-1})$  as  $\mu \rightarrow 0$ , whereas the previous choice for  $\beta$  gives that the right-hand side of (3.5) is  $O(\mu^{-\frac{3}{2}})$ . Therefore, choosing  $\mu$  small enough gives all the desired inequalities.

*Step 2: Time-dependent control.* We introduce a second comparison function which will help to control the solution  $u$  for small time. Let

$$\bar{u}(x, t) = At + C(1 - e^{-\gamma s}),$$

and write  $\psi(s) = 1 - e^{-\gamma s}$ . We will prove that for appropriate choices of  $A, C, \gamma, T^* > 0$ ,  $\bar{u}$  satisfies

$$\bar{u}_t - \mathcal{M}^-(D^2\bar{u}) \geq |D\bar{u}|^p \quad \text{in } \Gamma \times [0, T^*] \quad (3.6)$$

$$\bar{u} \geq u \quad \text{on } \partial_p(\Gamma \times [0, T^*]) \quad (3.7)$$

We first prove that  $\bar{u} \geq u$  on  $\partial_p(\Gamma \times [0, T^*])$ . Denote by  $\nu$  the exterior unit normal at  $x_0$ . For  $x = x_0 - s\nu, t = 0$  this is

$$\bar{u}(x, 0) = C(1 - e^{-\gamma s}) \geq u_0(x).$$

We use that

$$0 < \left| \frac{\partial \bar{u}}{\partial \nu}(x_0) \right| = C\psi'(0) = C\gamma < \infty, \quad (3.8)$$

$\|Du_0\|_\infty < \infty$ ,  $u_0(x_0) = \psi(x_0) = 0$ , and that both  $u_0, \bar{u}$  are non-negative to choose  $C > 0$  large enough so that for small  $s$ , say  $0 < s < \delta$ , we have

$$u_0(x) = u_0(x_0 - s\nu) \leq C\psi(s).$$

In other words, we are comparing the first-order expansions in the direction  $-\nu$ . Then, for  $\delta \leq s \leq \eta$ , we may also take

$$C \min\{1, \min_{\delta \leq s \leq \eta} \bar{u}(x_0 - s\nu, 0)\} > \max_{\Omega \times [0, T]} u_0, \quad (3.9)$$

since the minimum above is strictly positive. We may repeat this reasoning in the other directions which sweep  $\bar{\Gamma} \times \bar{\Omega}$ , by considering an extension by zero of  $u_0$  to the corresponding section of the annular domain where  $\psi$  is defined. The choice of  $C$  remains bounded since  $\bar{\Gamma} \cap \bar{\Omega}$  is compact, and furthermore, it remains uniform in  $x_0$  since  $\bar{\Omega}$  is compact. Observe that this also guarantees that  $\bar{u} \geq u$  on the rest of  $\partial_p(\Gamma \times [0, T^*])$ .

To check the equation, we compute

$$-\mathcal{M}^-(D^2\bar{u}) = \left( \Lambda\gamma - \lambda \frac{n-1}{s+\rho} \right) C\gamma e^{-\gamma s},$$

and observe that choosing  $\gamma$  large enough gives  $-\mathcal{M}^-(D^2\bar{u}) \geq 0$ . Then,  $\bar{u}$  is a supersolution if we can get

$$\bar{u}_t \geq |D\bar{u}|^p.$$

This amounts to taking

$$A \geq \max_{0 \leq s \leq \eta} (C\gamma e^{-\gamma s})^p.$$

We have therefore proved (3.6), and by comparison, it follows that  $\bar{u} \geq u$  in all of  $\Gamma \times [0, T^*]$ . Observe that we have not yet chosen the value of  $T^* > 0$ , hence the comparison just proved holds for arbitrary time. The comparison for small time will be done shortly.

*Step 3. Relating the comparison functions.* We will use the supersolution  $\bar{u}$  from the previous step to compare the solution  $u$  with the time-independent barrier  $\bar{v}$ , which has bounded gradient and satisfies the conditions on the lateral boundary.

We claim that for some  $T^* > 0$ ,

$$\bar{u}(x, t) \leq \bar{v}(x) \text{ for all } x \in \Gamma, 0 \leq t \leq T^*.$$

The proof is similar to that of the previous step. We establish first the comparison for  $t = 0$ , the bottom of the domain. Again we consider  $x = x_0 - s\nu$ . Recalling (3.8), we now seek

$$\left| \frac{\partial \bar{v}}{\partial \nu}(x_0) \right| = \varphi'(0) \geq C\gamma = \left| \frac{\partial \bar{u}}{\partial \nu}(x_0, 0) \right|. \quad (3.10)$$

From previous computations,

$$\varphi'(0) = \mu^{-\beta} \rightarrow +\infty \quad \text{as } \mu \rightarrow 0.$$

On the other hand,  $C$  depends on  $\mu$  through (3.9). Since  $\psi = \psi(s)$  is increasing inward, the minimum in (3.9) is achieved at  $s = \delta$ . Clearly we can take  $\delta < \mu = \eta$ , and so a simple computation shows

$$C = o(\mu^{-\beta}) \text{ as } \mu \rightarrow 0.$$

Hence taking  $\mu = \eta$  small enough eventually yields (3.10). We remark that this choice, which amounts to shrinking the domain  $\Gamma$ , does not affect the choices made for other constants.

As before, looking at the first order expansion gives  $\bar{u}(x, 0) \leq \bar{v}(x)$  for all  $x$  as above, near  $x_0$ , say with  $s < \delta'$ . Moreover, in this case it is easier to extend the inequality to the directions which sweep  $\Gamma \times \{t = 0\}$ , since both functions are radial and defined on the same annular domain.

Observe that to obtain

$$\bar{u}(x, 0) \leq \bar{v}(x) \quad \text{on } \partial\Gamma \cap \bar{\Omega} \quad (3.11)$$

there is no choice like (3.9) available. However, we may restrict the comparison to  $\Gamma_{\delta'} \times \{t = 0\}$ , where  $\Gamma_{\delta'} := \Gamma \cap \{\rho < s = |x - x_1| < \rho + \delta'\}$ ,  $\delta' > 0$  as above, so that (3.11) holds by comparing the first-order expansions, and more importantly, holds strictly. That is,  $\bar{u}(x, 0) < \bar{v}(x)$ . We may now take  $T^*$  so small that for  $0 \leq t \leq T^*$ ,

$$u(x, t) \leq \bar{u}(x, t) = At + \bar{u}(x, 0) \leq \bar{v}.$$

Thus we have proven that  $\bar{v}$  solves

$$\begin{aligned}\bar{v}_t - \mathcal{M}^-(D^2\bar{v}) &\geq |D\bar{v}|^p \quad \text{in } \Gamma_{\delta'} \times [0, T^*] \\ \bar{v} &\geq u \quad \text{on } (\partial\Gamma_{\delta'} \cap \bar{\Omega} \times [0, T^*]) \cup (\Gamma_{\delta'} \times \{t = 0\}) \\ \bar{v} &\geq 0 \quad \text{on } \Gamma_{\delta'} \cap \partial\Omega \times [0, T^*],\end{aligned}$$

Hence, by the comparison principle of [7] (see also Remark 2.2),  $u(x, t) \leq \bar{v}(x)$  in all of  $\bar{\Gamma}_{\delta'} \times [0, T^*]$ . In particular, this implies  $u(x_0, t) \leq \bar{v}(x_0) = 0$ , hence  $u(x_0, t) = 0$  for all  $0 \leq t \leq T^*$ . As  $x_0 \in \partial\Omega$  was arbitrary, this implies that the solution  $u$  satisfies the boundary conditions in the classical sense on  $\partial\Omega \times [0, T^*]$ .  $\square$

## 4 Regularization

In this section we apply the regularization procedure introduced in [22] to the viscosity solution of (1.1). By this procedure we obtain an equation satisfied in the pointwise a.e. sense, all of whose terms are integrable. For the sake of clarity, especially regarding notation, we recall some of the relevant definitions and properties, noting that we do not seek full generality in what follows.

**Definition 4.1.** For  $u \in C(\bar{\Omega} \times [0, T])$  and  $\epsilon, \kappa > 0$ , define

$$u_{\epsilon, \kappa}(x, t) = \inf_{(y, s) \in \Omega \times (0, T)} \left( u(y, s) + \frac{1}{2\epsilon}|x - y|^2 + \frac{1}{2\kappa}|t - s|^2 \right), \quad (4.1)$$

$$u^\epsilon(x, t) = \sup_{y \in \Omega} \left( u(y, t) - \frac{1}{2\epsilon}|x - y|^2 \right), \quad (4.2)$$

We may also define  $u^{\epsilon, \kappa}$  and  $u_\epsilon$  similarly. Note that we use just one index when the convolution is performed in the space variable only. In the following statement we collect a series of well-known facts regarding these operations which will be used shortly hereafter.

**Proposition 4.2.** Assume  $u \in C(\bar{\Omega} \times [0, T])$ , and let  $\epsilon, \kappa, \delta > 0$ .

(i) Both operations preserve both pointwise upper and lower bounds, i.e.,

$$\begin{aligned}\inf u &\leq u_{\epsilon, \kappa} \leq \sup u, \\ \inf u &\leq u^\epsilon \leq \sup u,\end{aligned}$$

where  $\inf$  and  $\sup$  are taken over  $\Omega \times (0, T)$ .

(ii) Let  $\epsilon^* = 2\sqrt{\epsilon\|u\|_\infty}$ ,  $\kappa^* = 2\sqrt{\kappa\|u\|_\infty}$ ,  $\Omega^{\epsilon^*} = \{x \in \Omega \mid d(x, \partial\Omega) > \epsilon^*\}$ . For all  $(x, t) \in \Omega^{\epsilon^*} \times (\kappa^*, 1 - \kappa^*)$ , there exist  $(y, s) \in \Omega \times (0, T)$  such that

$$u_{\epsilon, \kappa}(x, t) = u(y, s) + \frac{1}{2\epsilon}|x - y|^2 + \frac{1}{2\kappa}|t - s|^2.$$

In other words, the  $\sup$  and  $\inf$  in the definition of the convolutions are achieved, provided we are at a sufficient distance from the boundary.

- (iii) Both  $u_{\epsilon,\kappa}$  and  $u^{\epsilon,\kappa}$  are Lipschitz continuous in  $x$  with constant  $\frac{K}{\sqrt{\epsilon}}$ , where  $K = 2\|u\|_\infty$ . That is,

$$\sup_{\substack{x,y \in \Omega \\ t \in [0,T]}} \frac{|u(x,t) - u(y,t)|}{|x - y|} \leq \frac{K}{\sqrt{\epsilon}}.$$

Similarly, they are Lipschitz continuous in  $t$  with constant  $\frac{K}{\sqrt{\kappa}}$ .

- (iv)  $u^{\epsilon,\kappa}, u_{\epsilon,\kappa} \rightarrow u$  uniformly as  $\epsilon, \kappa \rightarrow 0$ , and similarly for  $u^\epsilon$ .  
(v)  $u^{\epsilon,\kappa}, u_{\epsilon,\kappa}$  are respectively semiconvex and semiconcave. In particular, they are twice differentiable a.e. That is, there are measurable functions  $a : \Omega \times [0, T] \mapsto \mathbb{R}$ ,  $q : \Omega \times [0, T] \mapsto \mathbb{R}^n$ ,  $M : \Omega \times [0, T] \mapsto S(n)$  such that

$$\begin{aligned} u^{\epsilon,\kappa}(y, s) &= u^{\epsilon,\kappa}(x, t) + a(x, t)(s - t) + \langle q(x, t), y - x \rangle \\ &\quad + \langle M(x, t)(y - x), y - x \rangle + o(|y - x|^2 + |s - t|). \end{aligned}$$

We will denote  $a = (u^{\epsilon,\kappa})_t$ ,  $q = Du^{\epsilon,\kappa}$ ,  $M = D^2u^{\epsilon,\kappa}$  for simplicity. The same goes for  $u_{\epsilon,\kappa}$ .

- (vi) With the notation above,

$$D^2u_{\epsilon,\kappa} \leq \frac{1}{\epsilon}I \quad \text{and} \quad D^2u^{\epsilon,\kappa} \geq -\frac{1}{\epsilon}I \quad \text{a.e. in } \Omega \times [0, T].$$

- (vii)  $(u_{\epsilon,\kappa})_\delta = u_{\epsilon+\delta,\kappa}$ .

- (viii)  $(u_{\epsilon+\delta,\kappa})^\delta \leq u_{\epsilon,\kappa}$ .

*Remark 4.3.* The easier proofs follow more or less directly from the definitions (see e.g., [13]), while (vii) and (viii) may be found in [11]. Property (v) uses the well-known theorems of Rademacher and Alexandrov on the differentiability of Lipschitz and convex functions, respectively; see [15] and the Appendix of [12].

The time-independent version of the following result appears as Lemma 4.2 in [28]. We say that  $F$  is proper if for all  $(X, \xi) \in S(n) \times \mathbb{R}^n$ ,  $r, s \in \mathbb{R}$ , if  $r \leq s$  then  $F(X, \xi, r) \leq F(X, \xi, s)$ .

**Lemma 4.4.** *Let  $u$  be a viscosity supersolution of  $u_t + F(D^2u, Du, u) = 0$  in  $\Omega \times (0, T)$ , where  $F$  is proper. Then, using the notation of Proposition 4.2,  $u_{\epsilon,\kappa}$  is a viscosity supersolution of  $u_t + F(D^2u, Du, u) = 0$  in  $\Omega^{\epsilon^*} \times (\kappa^*, T - \kappa^*)$ .*

*Proof:* Let  $\varphi = \varphi(x, t)$  be a  $C^2$  function that touches  $u_{\epsilon,\kappa}$  from below at  $(\hat{x}, \hat{t}) \in \Omega^{\epsilon^*} \times (\kappa^*, T - \kappa^*)$ , that is,  $\varphi(\hat{x}, \hat{t}) = u_{\epsilon,\kappa}(\hat{x}, \hat{t})$  and for  $|x - \hat{x}| + |t - \hat{t}| < \delta$  and sufficiently small  $\delta > 0$ ,

$$\varphi(x, t) \leq u_{\epsilon,\kappa}(x, t).$$

By Proposition 4.2, ii there exist  $(\hat{y}, \hat{s}) \in \Omega \times (0, T)$  such that

$$u_{\epsilon,\kappa}(\hat{x}, \hat{t}) = u(\hat{y}, \hat{s}) + \frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2 + \frac{1}{2\kappa}|\hat{t} - \hat{s}|^2.$$

Then, for  $(x, t)$  as above,

$$\varphi(x, t) \leq u_{\epsilon, \kappa}(x, t) \leq u(x + (\hat{y} - \hat{x}), t + (\hat{s} - \hat{t})) + \frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2 + \frac{1}{2\kappa}|\hat{t} - \hat{s}|^2.$$

Evaluating this expression now at  $(x + (\hat{x} - \hat{y}), t + (\hat{t} - \hat{s}))$ , we have that

$$\tilde{\varphi}(x, t) := \varphi(x + (\hat{x} - \hat{y}), t + (\hat{t} - \hat{s})) - \frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2 - \frac{1}{2\kappa}|\hat{t} - \hat{s}|^2 \leq u(x, t)$$

We also have, from the choice of  $(\hat{y}, \hat{s})$ , that  $\tilde{\varphi}(\hat{y}, \hat{s}) = u(\hat{y}, \hat{s})$ . Hence  $\tilde{\varphi}$  is a valid test function for  $u$ . Since

$$\begin{aligned} D^2\tilde{\varphi}(\hat{y}, \hat{s}) &= D^2\varphi(\hat{x}, \hat{t}), & D\tilde{\varphi}(\hat{y}, \hat{s}) &= D\varphi(\hat{x}, \hat{t}), \\ \tilde{\varphi}_t(\hat{y}, \hat{s}) &= \varphi_t(\hat{x}, \hat{t}), & \tilde{\varphi}(\hat{y}, \hat{s}) &\leq \varphi(\hat{x}, \hat{t}), \end{aligned}$$

by the properness of  $F$ ,

$$\begin{aligned} \varphi_t(\hat{x}, \hat{t}) + F(D^2\varphi(\hat{x}, \hat{t}), D\varphi(\hat{x}, \hat{t}), \varphi(\hat{x}, \hat{t})) &\geq \\ \tilde{\varphi}_t(\hat{y}, \hat{s}) + F(D^2\tilde{\varphi}(\hat{y}, \hat{s}), D\tilde{\varphi}(\hat{y}, \hat{s}), \tilde{\varphi}(\hat{y}, \hat{s})) &\geq 0. \end{aligned}$$

Hence,  $u_{\epsilon, \kappa}$  is a supersolution. That it is a subsolution is proved similarly.  $\square$

**Proposition 4.5.** *Let  $\Omega' \subset \subset \Omega$ ,  $0 < t_0 < t_1 < T$  and  $u$  be a viscosity supersolution of (1.1) in  $\Omega \times (0, T)$ . Then, there exist constants  $\epsilon, \delta, \kappa > 0$  such that the regularized function  $w = (u_{\epsilon+\delta, \kappa})^\delta$  satisfies*

$$w_t - \mathcal{M}^-(D^2w) - |Dw|^p \geq 0 \quad \text{a.e. in } \Omega' \times (t_0, t_1). \quad (4.3)$$

*In particular,  $w$  is an  $L^\infty$ -strong supersolution of (4.3).*

*Proof:* We apply Lemma 4.4 to (1.1) to find that, for sufficiently small  $\epsilon$  and  $\kappa$ ,  $u_{\epsilon, \kappa}$  is a viscosity supersolution of (1.1) in  $\Omega' \times (t_0, t_1)$ . Observe that the Lemma applies since there is no  $x$  dependence. The regularized function  $w$  defined above is both semiconvex and semiconcave in  $x$ , as well as Lipschitz-continuous in  $t$ . Hence it is twice differentiable a.e. in  $\Omega' \times (\tau, 1 - \tau)$ , in the sense of having a second order “parabolic” Taylor expansion (as in Proposition 4.2, (v)).

Let  $(\hat{x}, \hat{t})$  be any such point of differentiability. As in Proposition 4.2, (viii), we have that  $w \leq u_{\epsilon, \kappa}$ . Suppose that  $w(\hat{x}, \hat{t}) = u_{\epsilon, \kappa}(\hat{x}, \hat{t})$ . For  $(x, t)$  in a neighborhood of  $(\hat{x}, \hat{t})$ , we then have

$$\begin{aligned} u_{\epsilon, \kappa}(x, t) &\geq w(x, t) = w(\hat{x}, \hat{t}) + w_t(\hat{x}, \hat{t})(t - \hat{t}) + \langle Dw(\hat{x}, \hat{t}), x - \hat{x} \rangle \\ &\quad + \langle D^2w(\hat{x}, \hat{t}), x - \hat{x} \rangle + o(|x - \hat{x}|^2 + |t - \hat{t}|) \\ &= u_{\epsilon, \kappa}(\hat{x}, \hat{t}) + w_t(\hat{x}, \hat{t})(t - \hat{t}) + \langle Dw(\hat{x}, \hat{t}), x - \hat{x} \rangle \\ &\quad + \langle D^2w(\hat{x}, \hat{t}), x - \hat{x} \rangle + o(|x - \hat{x}|^2 + |t - \hat{t}|), \end{aligned}$$

which implies that  $(w_t(\hat{x}, \hat{t}), Dw(\hat{x}, \hat{t}), D^2w(\hat{x}, \hat{t})) \in \mathcal{P}^{2,-}u_{\epsilon, \kappa}(\hat{x}, \hat{t})$ , the parabolic subject at  $(\hat{x}, \hat{t})$  (see, e.g., [12]). Since  $u_{\epsilon, \kappa}$  is a viscosity supersolution, this gives

$$w_t(\hat{x}, \hat{t}) - \mathcal{M}^-(D^2w(\hat{x}, \hat{t})) - |Dw(\hat{x}, \hat{t})|^p \geq 0.$$



Assume now that  $w(\hat{x}, \hat{t}) < u_{\epsilon, \kappa}(\hat{x}, \hat{t})$ . In this case, by Proposition 4.4 in [11],  $D^2w(\hat{x}, \hat{t})$  has an eigenvalue equal to  $-\frac{1}{\delta}$ . On the other hand, by Proposition 4.5 in [11],  $w$  is  $\frac{1}{2\epsilon}$ -semiconvex, so the remaining eigenvalues are bounded by above by  $\frac{1}{\epsilon}$ . Recalling also the gradient bounds which come from the Lipschitz continuity of  $w$  with respect to both  $x$  and  $t$ , as in Proposition 4.2, (iii), we obtain

$$w_t(\hat{x}, \hat{t}) - \mathcal{M}^-(D^2w(\hat{x}, \hat{t})) - |Dw(\hat{x}, \hat{t})|^p \geq -\frac{K}{\kappa^{\frac{1}{2}}} + \lambda \frac{1}{\delta} - (n-1)\Lambda \frac{1}{\epsilon} - \frac{K^p}{\epsilon^{\frac{p}{2}}}.$$

By taking  $\delta = o(\epsilon^{\frac{p}{2}})$  and  $\epsilon$  sufficiently small, the right-hand side of the above inequality becomes nonnegative. Hence,  $w$  is a supersolution.  $\square$

*Remark 4.6.* We will apply Lemma 4.5 in the case where  $\Omega = (0, 1)$ , and regularization is applied to  $U = U(r)$ , the radial part of the solution  $u$  of (1.1) in  $B_1(0)$ . The spatial regularization is performed with respect to the radial variable. Precisely, we have

$$\begin{aligned} w(r, t) &= (U_{\tilde{\epsilon} + \delta, \kappa})^\delta(r, t) = \\ &= \sup_{r_1 \in (0, 1)} \inf_{\substack{r_2 \in (0, 1) \\ s \in (0, T)}} \left( U(r_2, s) + \frac{1}{2(\tilde{\epsilon} + \delta)} |r_2 - r_1|^2 + \frac{1}{2\kappa} |t - s|^2 - \frac{1}{2\delta} |r - r_1|^2 \right). \end{aligned}$$

We have changed notation slightly, using  $\tilde{\epsilon}$  for the spatial regularization parameter for convenience. Note also that from the proof of Lemma 4.5, we choose  $\delta = \delta(\tilde{\epsilon})$ , with  $\delta \rightarrow 0$  as  $\tilde{\epsilon} \rightarrow 0$ , so we need only choose suitable  $\tilde{\epsilon} > 0$  in the regularization.

From the Lemma we conclude that  $w$  satisfies (4.3) in  $(\epsilon, 1 - \epsilon) \times (t_0, t_1)$  for arbitrary  $\epsilon > 0$ , provided we choose a small enough  $\tilde{\epsilon} > 0$ . For this reason, we are able to take arbitrarily small  $\epsilon > 0$  in the proof of Theorem 1.2.

## 5 Radial form

**Lemma 5.1.** *Let  $u \in C(\overline{\Omega})$  be the viscosity solution of*

$$\begin{aligned} u_t - \mathcal{M}^-(D^2u) - |Du|^p &= 0 \quad \text{in } B_1(0) \times [0, T], \\ u &= 0 \quad \text{on } \partial B_1(0) \times [0, T], \\ u(\cdot, 0) &= u_0 \quad \text{in } B_1(0). \end{aligned}$$

where  $u_0$  is a radial function. Then  $u$  is radial as well, that is,  $u(x) = U(|x|)$  for some  $U : [0, 1] \times [0, T] \mapsto \mathbb{R}^n$ , and  $U$  solves

$$\begin{aligned} U_t - \theta(U'')U'' - \frac{n-1}{r}\theta(U')U' - |U'|^p &= 0 \quad \text{in } (0, 1) \times (0, T), \\ U &= 0 \quad \text{on } \{r = 1\} \times [0, T], \\ U(\cdot, 0) &= u_0 \quad \text{in } B_1(0), \end{aligned} \tag{5.1}$$

in the viscosity sense, where  $'$  denotes the radial derivative and

$$\theta(s) = \begin{cases} \lambda, & \text{if } s > 0, \\ \Lambda, & \text{if } s \leq 0. \end{cases}$$

*Proof:* For a  $C^2$  radial function, say  $\phi(x) = \Phi(|x|)$ , a standard computation of the eigenvalues of  $D^2\phi$  at any point gives them explicitly as  $\Phi''$  and  $\frac{\Phi'}{r}$ , the latter with multiplicity  $n - 1$ . By the definition of the extremal operator, this gives

$$\mathcal{M}^-(D^2\phi) = \theta(\Phi'')\Phi'' + \frac{n-1}{r}\theta(\Phi')\Phi'. \quad (5.2)$$

That  $u$  is radial follows from the uniqueness of solutions of the Cauchy-Dirichlet problem, the rotation-invariance of the equation and the fact that the initial data is radial. Hence, there exists a  $U$  as described above. We now prove that it is subsolution of (5.1). Consider  $\Phi((0, 1) \times [0, T]) \in C^2$  that touches  $U$  from above at  $(\hat{r}, \hat{t})$ . Define  $\phi(x, t) = \Phi(|x|, t)$ . The lemma follows from applying (5.2) to  $\phi$ , which is  $C^2$ , radial and a valid test function for  $u$  at any  $(\hat{x}, \hat{t})$  such that  $|\hat{x}| = \hat{r}$ , observing also that  $|D\phi| = |\Phi'|$ . The proof that  $U$  is a supersolution as well is analogous.  $\square$

*Remark 5.2.* In what follows we will at times write simply  $u(x) = u(r)$  for radial functions, as is standard. This notation was avoided in the last lemma for clarity.

*Remark 5.3.* From the radial equation obtained above, we can obtain an inequality in divergence form as follows. Combining Proposition 4.5, Lemma 5.1, and the considerations of Remark 4.6,  $w$  satisfies

$$w_t - \theta(w'')w'' - \frac{n-1}{r}\theta(w')w' - |w'|^p \geq 0$$

for a.e.  $r \in (\epsilon, 1 - \epsilon), t \in (t_0, t_1)$ ,

for arbitrary choices of  $\epsilon > 0$  and  $0 < t_0 < t_1 < T$ . Multiplying by the continuous, non-negative weight  $\tilde{\rho}(r) = \frac{\rho}{\theta(w'')}$ ,  $r \in (\epsilon, 1 - \epsilon)$ , where

$$\rho(r) = e^{\int_{1-\epsilon}^r \frac{\hat{n}-1}{s} ds},$$

$$\hat{n} = \frac{\theta(w')}{\theta(w'')}(n-1) + 1,$$

we have, for the second-order terms,

$$\tilde{\rho} \left( \theta(w'')w'' + \frac{n-1}{r}\theta(w')w' \right) = (\rho w')'.$$

Hence,

$$\tilde{\rho} w_t \geq (\rho w')' + \tilde{\rho} |w'|^p \quad \text{for a.e. } r \in (\epsilon, 1 - \epsilon), t \in (t_0, t_1). \quad (5.3)$$

*Remark 5.4.* The weight  $\tilde{\rho}$  depends on the regularization parameters both explicitly and through the solution of the approximate equation  $w$ , but we omit these dependencies for simplicity of notation. However, as

$$\hat{n} - 1 = \frac{\theta(w')}{\theta(w'')}(n - 1) \leq \frac{\Lambda}{\lambda}(n - 1),$$

we have for  $\epsilon \in (0, \frac{1}{2})$  and all  $r \in (\epsilon, 1 - \epsilon)$ , that

$$\begin{aligned} \tilde{\rho}(r) &= \frac{1}{\theta(w'')} e^{\int_{1-\epsilon}^r \frac{\hat{n}-1}{s} ds} \\ &\geq \frac{1}{\Lambda} \left( \frac{r}{1-\epsilon} \right)^{\frac{\Lambda}{\lambda}(n-1)} \\ &\geq \frac{1}{\Lambda} \left( \frac{r}{2} \right)^{\frac{\Lambda}{\lambda}(n-1)} := \hat{\rho}(r). \end{aligned} \tag{5.4}$$

Note that  $\hat{\rho}$  no longer depends on the regularization parameters and is defined for all  $r \in (0, 1)$ . On the other hand, since  $\frac{r}{1-\epsilon} \leq 1$  for all  $r \in (\epsilon, 1 - \epsilon)$ , we similarly obtain

$$\tilde{\rho}(r) \leq \frac{1}{\lambda}. \tag{5.5}$$

Note that this bound is also independent of the regularization parameters.

## 6 Nonexistence of global solutions and LOBC

The proof of the radial case of our main theorem involves the solution to the Dirichlet eigenvalue problem for the extremal operator  $-\mathcal{M}^-$  in annular domains approximating the punctured ball  $B_1(0) \setminus \{0\}$ .

More precisely, consider

$$\begin{aligned} -\mathcal{M}^-(D^2\varphi) &= \lambda\varphi && \text{in } A_\epsilon, \\ \varphi &= 0 && \text{on } \partial A_\epsilon. \end{aligned} \tag{6.1}$$

where  $A_\epsilon = B_{1-\epsilon}(0) \setminus \overline{B_\epsilon(0)}$ . Note that this corresponds to the spatial domain where (5.3) is satisfied.

By Proposition 1.1 in [9], there exists a solution pair  $(\lambda_1^\epsilon, \varphi_1^\epsilon)$  of (6.1) with  $\lambda_1^\epsilon > 0$ ,  $\varphi_1^\epsilon \in C^2(A_\epsilon) \cap C(\overline{A_\epsilon})$  and  $\varphi_1^\epsilon > 0$  in  $A_\epsilon$ , where  $\varphi_1^\epsilon$  is unique up to a positive constant. We normalize this solution so that  $\varphi_1^\epsilon(\frac{1}{2}) = 1$ , for reasons that will become apparent later.

Our notation indicates that both  $\lambda_1^\epsilon$  and  $\varphi_1^\epsilon$  depend on the parameters of the spatial regularization through the domain  $A_\epsilon$ . We will omit these dependencies in the notation for simplicity whenever possible, but stress that they are relevant later in the proof and will be taken into account.

We employ the following lemma to state our main theorem without reference to these regularization parameters.

**Lemma 6.1.** *Let  $K \subset (0, 1)$  be a closed interval such that  $\frac{1}{2} \in K$ . There exists a function  $\hat{\varphi} \in C(K)$ , such that  $\hat{\varphi}(r) > 0$  for all  $r \in K$  and  $\varphi_1^\epsilon \rightarrow \hat{\varphi}$  uniformly over  $K$ .*

*Proof.* In general, if we denote by  $\lambda_1(\Omega)$  the corresponding *principal half-eigenvalue* (i.e., solution of (6.1)) in  $\Omega$ , we have that  $\lambda_1(\Omega') \leq \lambda_1(\Omega)$  if  $\Omega \subset \Omega'$ ; see Proposition 1.1 (iii) in [9]. We therefore have the monotonicity  $\lambda_1^{\epsilon'} \leq \lambda_1^\epsilon$  if  $\epsilon' \leq \epsilon$ , and for the same reason,  $\lambda_1(B_1(0)) \leq \lambda_1^\epsilon$  for all  $\epsilon > 0$ . Then,  $\lambda_1^\epsilon \rightarrow \hat{\lambda}$  as  $\epsilon \rightarrow 0$ , for some  $\hat{\lambda} > 0$ .

Consider now a closed interval  $K' \supset K$ . By Harnack's inequality (see Theorem 3.6 in [26]), for all  $\epsilon > 0$  small enough such that  $K' \subset A_\epsilon$ , we have

$$\sup_{K'} \varphi_1^\epsilon \leq \sup_{A_\epsilon} \varphi_1^\epsilon \leq C \inf_{A_\epsilon} \varphi_1^\epsilon \leq C \inf_{K'} \varphi_1^\epsilon \leq C,$$

where we used  $\varphi_1^\epsilon(\frac{1}{2}) = 1$  and  $\frac{1}{2} \in K$  for the last inequality.

It follows that the functions  $\varphi_1^\epsilon$  satisfy a family of ODEs (which come from the radial expression of the extremal operator) with uniformly bounded right-hand sides

$$\begin{cases} F((\varphi_1^\epsilon)'', (\varphi_1^\epsilon)', r) := \mathcal{M}^-(D^2 \varphi_1^\epsilon) = -\lambda_1^\epsilon \varphi_1^\epsilon \leq -\hat{\lambda}C & \text{in int}(K'), \\ \varphi_1^\epsilon \geq 0 & \text{on } \partial K'. \end{cases}$$

We can then use a compactness lemma (Lemma 3.1 in [14]) to conclude that

$$\varphi_1^\epsilon(r) \leq C \text{ and } (\varphi_1^\epsilon)'(r) \leq C$$

for some  $C > 0$  and all  $r \in K'$ . Then, there exists a subsequence  $\varphi_1^{\epsilon_k} \rightarrow \hat{\varphi}$  for some  $\hat{\varphi} \in C(K')$ . We recall that  $\lambda_1^{\epsilon_k} \rightarrow \hat{\lambda}$ . Therefore, by stability of viscosity solutions, we pass to the limit in equation (6.1) (restricted to  $\text{int}(K')$ ) to obtain

$$-\mathcal{M}^-(D^2 \hat{\varphi}) = \hat{\lambda} \hat{\varphi} \text{ in int}(K').$$

From the uniform convergence and  $\varphi_1^\epsilon \geq 0$  for all  $\epsilon > 0$ , we conclude  $\hat{\varphi} \geq 0$ , hence the strong maximal principle applies (see Lemma 3.4 in [19]). Recalling also the normalization, we have  $\hat{\varphi}(\frac{1}{2}) = 1$ , hence  $\hat{\varphi} \not\equiv 0$ . Combining these facts, we conclude that  $\hat{\varphi} > 0$  in  $\text{int}(K') \supset K$ .  $\square$

*Remark 6.2.* For any compact  $K$ ,  $\hat{\varphi}$  is bounded by below on  $K$  by a positive constant. By uniform convergence, this implies that all the  $\varphi_1^\epsilon$  are bounded by below on any such  $K$  by a positive constant, uniformly in  $\epsilon$ .

The proof of Theorem 1.2 requires two additional lemmas.

**Lemma 6.3.** *Let  $\varphi_1^\epsilon$  be the solution of (6.1), as defined above. Then, for any  $0 < \alpha < 1$ , there exists a positive constant  $C > 0$  such that*

$$\int_\epsilon^{1-\epsilon} (\varphi_1^\epsilon)^{-\alpha} \tilde{\rho} dr < C. \quad (6.2)$$

Furthermore,  $C$  may be taken uniformly for  $\epsilon \in (0, \frac{1}{4})$ .

*Proof.* By Remark 6.2, it is possible to bound  $\varphi_1^\epsilon$  by below by a positive constant uniformly for small  $\epsilon$  over a closed interval  $K \subset (0, 1)$ , to be chosen later. Hence, to obtain (6.2) it is sufficient to bound the integral near the endpoints  $\epsilon$  and  $1 - \epsilon$ .

We now proceed as in the proof of Hopf's lemma to obtain a uniform lower bound for  $\varphi_1'(\epsilon)$ . Let  $\beta > 0$  and

$$v(r) = \frac{e^{-\beta(\frac{1}{2}-r)^2} - e^{-\beta(\frac{1}{2}-\epsilon)^2}}{1 - e^{-\beta(\frac{1}{2}-\epsilon)^2}} \quad \text{for } \epsilon < r < \frac{1}{2}.$$

We verify that  $v \geq 0$ ,  $v(\epsilon) = 0$ ,  $v(\frac{1}{2}) = 1$ , and compute

$$v'(r) = \frac{2\beta(\frac{1}{2}-r)e^{-\beta(\frac{1}{2}-r)^2}}{1 - e^{-\beta(\frac{1}{2}-\epsilon)^2}} \geq 0, \quad (6.3)$$

$$v''(r) = \frac{(4\beta^2(\frac{1}{2}-r)^2 - 2\beta)e^{-\beta(\frac{1}{2}-r)^2}}{1 - e^{-\beta(\frac{1}{2}-\epsilon)^2}} \geq 0, \quad (6.4)$$

where the inequality in (6.4) follows from taking a sufficiently large  $\beta > 0$ .

We abuse notation slightly and define  $v(x) = v(|x|)$  in  $B_{\frac{1}{2}}(0) \setminus \overline{B_\epsilon(0)}$ . By the previous computation, using also that  $\lambda_1^\epsilon > 0$ ,  $v \geq 0$ , we have

$$\mathcal{M}^-(D^2v) = \lambda v'' + \lambda \frac{n-1}{r} v' \geq 0 \geq -\lambda_1^\epsilon v.$$

Hence  $v$  is a subsolution of (6.1). Since  $v(\epsilon) = \varphi_1^\epsilon(\epsilon)$ ,  $v(\frac{1}{2}) = \varphi_1^\epsilon(\frac{1}{2})$ , by comparison we have  $v(r) \leq \varphi_1^\epsilon(r)$  for all  $\epsilon < r < \frac{1}{2}$  (see, for example, Appendix A in [2]). Recalling (6.3), for all  $0 < \epsilon < \frac{1}{4}$  this gives

$$(\varphi_1^\epsilon)'(\epsilon) \geq v'(\epsilon) = \frac{2\beta(\frac{1}{2}-\epsilon)e^{-\beta(\frac{1}{2}-\epsilon)^2}}{1 - e^{-\beta(\frac{1}{2}-\epsilon)^2}} \geq \frac{\beta e^{-\frac{\beta}{4}}}{1 - e^{-\frac{\beta}{4}}} =: C$$

Note that the last constant does not depend on  $\epsilon$ .

By looking at the first order expansion of  $\varphi_1$  at  $\epsilon$ ,

$$\varphi_1(r) = (\varphi_1)'(\epsilon)(r - \epsilon) + o(|r - \epsilon|),$$

we have that there exists a  $\delta > 0$  such that

$$\varphi_1(r) > \frac{(\varphi_1)'(\epsilon)}{2}(r - \epsilon) \quad \text{for all } \epsilon < r < \epsilon + \delta.$$

In the above series expansion, the constant  $\delta > 0$  depends only  $(\varphi_1)'(\epsilon)$ , which is uniformly bounded by the compactness lemma of [14]. Hence,  $\delta > 0$  does not depend on  $\epsilon$ .

We then estimate, for  $\epsilon < r < \epsilon + \delta$ ,

$$\begin{aligned} (\varphi_1(r))^{-\alpha} &< \left( \frac{(\varphi_1)'(\epsilon)}{2} \right)^{-\alpha} (r - \epsilon)^{-\alpha} \\ &\leq C^{-\alpha} (r - \epsilon)^{-\alpha}. \end{aligned}$$

Recall the bound  $\tilde{\rho}(r) \leq \frac{1}{\lambda}$  given by (5.5). Then,

$$\begin{aligned} \int_{\epsilon}^{\epsilon+\delta} (\varphi_1(r))^{-\alpha} \tilde{\rho} dr &\leq C^{-\alpha} \|\tilde{\rho}\|_{\infty} \int_{\epsilon}^{\epsilon+\delta} (r-\epsilon)^{-\alpha} dr \\ &\leq C^{-\alpha} \frac{1}{\lambda} \int_0^{\delta} r^{-\alpha} dr < \tilde{C}. \end{aligned}$$

A similar bound can be obtained over  $1-\epsilon-\delta < r < 1-\epsilon$ . We may then choose  $K = [\delta, 1-\delta]$  with  $\delta$  as above and recall that  $\varphi_1^{\epsilon}$  are uniformly bounded by below on  $K$ . Also note that for all  $\epsilon > 0$ ,

$$K = [\delta, 1-\delta] \supset (\epsilon+\delta, 1-\epsilon-\delta), \quad (6.5)$$

hence, we may combine the bounds near the endpoints with the lower bound on  $K$  to obtain (6.2).  $\square$

*Remark 6.4.* The interval  $K = [\delta, 1-\delta]$  is the one that appears in the statement of Theorem 1.2. As claimed,  $\delta$  depends only on  $\lambda, \Lambda, n$ .

**Lemma 6.5.** *Consider  $\tilde{\rho} : [0, 1] \rightarrow \mathbb{R}$  as defined above. There exists a constant  $C > 0$  such that*

$$\int_0^1 |w| \tilde{\rho} dx \leq C \int_0^1 |w'| \tilde{\rho} dx \quad (6.6)$$

for all  $w \in C^1(0, 1)$ , such that  $w(1) = 0$ .

In the proof of the above inequality we employ the following result from [31]:

**Theorem 6.6.** *Let  $\nu : [0, 1] \rightarrow \mathbb{R}$  be a non-negative, non-vanishing, continuous weight on the closed unit interval. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be once differentiable and satisfy  $f(0) = 0$ . Then,*

$$\int_0^1 |f(x)| \nu(x) dx \leq \left( \max_{0 \leq x \leq 1} \frac{1}{\nu(x)} \int_x^1 \nu(z) dz \right) \int_0^1 |f'(x)| \nu(x) dx, \quad (6.7)$$

and the constant is sharp.

*Proof of Lemma 6.5:* Let  $v$  be once differentiable such that  $v(1) = 0$ , let  $\delta > 0$ , and define

$$f(x) = v(1-x), \quad \nu(x) = \tilde{\rho}(1-x) + \delta.$$

As  $\tilde{\rho}$  is continuous and non-negative on  $[0, 1]$ ,  $f$  and  $\nu$  so defined satisfy the hypothesis of the theorem. After changing variables in the integrals involving  $f$  only, (6.7) reads

$$\int_0^1 |v(x)| (\tilde{\rho}(x) + \delta) dx \leq C_{\delta} \int_0^1 |v'(x)| (\tilde{\rho}(x) + \delta) dx \quad (6.8)$$

where

$$C_{\delta} = \max_{0 \leq x \leq 1} \frac{1}{\tilde{\rho}(1-x) + \delta} \int_x^1 \tilde{\rho}(1-z) + \delta dz.$$

The weight  $\tilde{\rho}$  is absolutely continuous, and a simple computation shows  $\tilde{\rho}'(x) > 0$  for a.e.  $x \in [0, 1]$ . Hence,  $\tilde{\rho}(1 - x)$  is non-increasing and

$$\begin{aligned} \frac{1}{\tilde{\rho}(1 - x) + \delta} \int_x^1 \tilde{\rho}(1 - z) + \delta \, dz &= \frac{1}{\tilde{\rho}(1 - x) + \delta} (\tilde{\rho}(1 - x) + \delta)(1 - x) \\ &\leq (1 - x) \leq 1. \end{aligned}$$

Thus, taking  $\delta \rightarrow 0$  in (6.8) gives

$$\int_0^1 |v(x)| \tilde{\rho}(x) \, dx \leq \int_0^1 |v'(x)| \tilde{\rho}(x) \, dx,$$

for once differentiable  $v$  with  $v(1) = 0$ .  $\square$

*Remark 6.7.* Lemma 6.5 is valid on an arbitrary interval. For any interval contained in  $(0, 1)$  we still have that the constant is bounded by 1.

We now prove our main result in the radial case, i.e., when the spatial domain corresponds to a ball and the initial data is radially symmetric. The proof of the result in a general domain follows more or less easily from the radial case.

*Proof of Theorem 1.2:* Our proof uses key ideas from that of Theorem 2.1 in [29]. Some care is required in choosing the constants appearing in our argument in the correct order; specifically in choosing  $u_0$  large in an appropriate sense while choosing the regularization parameters sufficiently small. This difficulty is not present in [29], since the solutions dealt with therein are classical and no regularization is needed.

Consider the differential inequality

$$\dot{y} \geq \frac{1}{2} y^p, \quad t_0 < t < t_1 \quad (6.9)$$

$$y(t_0) = M_0, \quad (6.10)$$

where  $0 < t_0 < t_1$ , and  $M_0 > 0$ . We can integrate (6.9) explicitly to obtain

$$0 \leq y(t)^{1-p} \leq (1-p) \left( \frac{1}{2}(t - t_0) + \frac{y_0^{1-p}}{1-p} \right).$$

Therefore,  $y(t)^{1-p} \rightarrow 0$  as  $t \rightarrow t_0 + \frac{2M_0^{1-p}}{(p-1)}$ . Since  $1 - p < 0$ , this implies  $y(t) \rightarrow +\infty$ . Alternatively, for a fixed  $t_1 > t_0$ , blow-up occurs for  $t < t_1$  provided  $M_0 > \left[ \frac{p-1}{2}(t_1 - t_0) \right]^{-\frac{1}{p-1}}$ .

So fix  $T > 0$  and assume there exists a solution  $u$  of (1.1) in  $B_1(0) \times [0, T]$  that satisfies (1.2) in the classical sense, with radial initial data  $u_0 \in C^1(\overline{B_1(0)})$ . We will later specify the largeness condition on  $u_0$ , but may consider  $\|u_0\|_\infty$  fixed from now on.

Recall now the regularized function  $w$  obtained in Lemma 4.5, which in the radial case is a solution of the inequality (5.3) in  $A_\epsilon = B_1(0) \setminus \overline{B_\epsilon(0)}$ . We take  $\epsilon$  so that  $0 < \epsilon < \delta$  where  $\delta$  is the constant given by Lemma 6.3, which also appears in the statement of Theorem 1.2 (see Remark 6.4); i.e.,  $\epsilon$  is small enough so that  $K = [\delta, 1 - \delta] \subset (\epsilon, 1 - \epsilon)$ . Consider also the corresponding solution pair  $(\lambda_1, \varphi_1)$  in  $A_\epsilon$ . Define

$$z(t) = \int_\epsilon^{1-\epsilon} w(r, t) \varphi_1(r) \tilde{\rho} dr.$$

We proceed to show that  $z = z(t)$  satisfies (6.9) with large initial data, and hence blows up for some  $t < t_1$ . This is a contradiction, since  $z$  is uniformly bounded for all  $t > 0$  by the uniform convergence of  $w \rightarrow u$ , the fact that  $u$  is uniformly bounded (see Remark 2.6), and the bounds for  $\hat{\varphi}$  and  $\hat{\rho}$ . Therefore, the solution  $u$  that satisfies the boundary data in the classical sense must cease to exist. In other words, LOBC occurs.

Using (5.3), we compute

$$\begin{aligned} \dot{z}(t) &= \int_\epsilon^{1-\epsilon} w_t(r, t) \varphi_1(r) \tilde{\rho} dr \\ &\geq \int_\epsilon^{1-\epsilon} ((\rho w')' + |w'|^p) \varphi_1(r) \tilde{\rho} dr \\ &= \int_\epsilon^{1-\epsilon} (\rho w')' \varphi_1(r) \tilde{\rho} dr + \int_\epsilon^{1-\epsilon} |w'|^p \varphi_1(r) \tilde{\rho} dr \\ &=: I_1 + I_2 \end{aligned}$$

Next we integrate by parts twice in  $I_1$ . In the first integration, boundary terms are null since  $\varphi_1(\epsilon) = \varphi_1(1 - \epsilon) = 0$ . Then,

$$\begin{aligned} I_1 &= - \int_\epsilon^{1-\epsilon} \rho w' \varphi_1' dr \\ &= \int_\epsilon^{1-\epsilon} w (\rho \varphi_1')' dr + w \rho \varphi_1'|_\epsilon^{1-\epsilon}. \end{aligned}$$

We know from the compactness lemma in [14] that

$$-C \leq \varphi_1'(\epsilon) < 0 < \varphi_1'(1 - \epsilon) \leq C,$$

and from the definition of the weight  $\tilde{\rho}$  and the bound (5.5), that

$$0 \leq \rho(\epsilon) \rightarrow 0, \quad 0 \leq \rho(1 - \epsilon) \leq C$$

as  $\epsilon \rightarrow 0$ , where  $C > 0$  above is independent of  $\|u_0\|_\infty$  and of the approximation parameters. Using the uniform convergence  $w \rightarrow u$ , the continuity of  $u$ , and  $u(\cdot, t)|_{\partial\Omega} = 0$  for all  $t \in [0, T]$ , we have that

$$0 \leq w(1 - \epsilon, t) \rightarrow 0, \quad 0 \leq w(\epsilon, t) = O(\|u_0\|_\infty), \quad (6.11)$$



again as  $\epsilon \rightarrow 0$ . For the second of these inequalities we are also using the fact that  $\|u\|_\infty \leq \|u_0\|_\infty$  (see Remark 2.6).

Therefore,

$$w\rho\varphi_1'|_\epsilon^{1-\epsilon} \geq -c_1\|u_0\|_\infty, \quad (6.12)$$

where  $0 < c_1 \rightarrow 0$  as  $\epsilon \rightarrow 0$ .

We continue estimating

$$\begin{aligned} I_1 &\geq \int_\epsilon^{1-\epsilon} w(\rho\varphi_1')' dr - c_1\|u_0\|_\infty \\ &\geq \int_\epsilon^{1-\epsilon} w(\tilde{\rho}\mathcal{M}^-(D^2\varphi_1)) dr - c_1\|u_0\|_\infty \\ &= \int_\epsilon^{1-\epsilon} w(-\lambda_1\varphi_1)\tilde{\rho} dr - c_1\|u_0\|_\infty \\ &= -\lambda_1 z(t) - c_1\|u_0\|_\infty. \end{aligned}$$

The inequality in the second line above comes from the minimality of the Pucci operator: for all  $\varphi \in C^2$ ,  $\tilde{\rho}\mathcal{M}^-(D^2\varphi) \leq (\rho\varphi')'$ . We have then obtained

$$\dot{z}(t) + \lambda_1 z(t) \geq I_2 - c_1\|u_0\|_\infty. \quad (6.13)$$

We turn to estimating  $I_2$ . From Hölder's inequality for the measure  $\tilde{\rho}(r) dr$ , we get

$$\begin{aligned} \int_\epsilon^{1-\epsilon} |w'| \tilde{\rho} dr &= \int_\epsilon^{1-\epsilon} |w'| (\varphi_1)^{\frac{1}{p}} (\varphi_1)^{-\frac{1}{p}} \tilde{\rho} dr \\ &\leq \left( \int_\epsilon^{1-\epsilon} (\varphi_1)^{-\frac{1}{p-1}} \tilde{\rho} dr \right)^{\frac{p-1}{p}} \left( \int_\epsilon^{1-\epsilon} |w'|^p \varphi_1 \tilde{\rho} dr \right)^{\frac{1}{p}} \\ &\leq C \left( \int_\epsilon^{1-\epsilon} |w'|^p \varphi_1 \tilde{\rho} dr \right)^{\frac{1}{p}} \\ &= CI_2^{\frac{1}{p}} \end{aligned}$$

where  $C$  above is finite and does not depend on  $\epsilon$ , by Lemma 6.3.

Next we apply Lemma 6.5 to  $\tilde{w}(r, t) := w(r, t) - w(1 - \epsilon, t)$  on the interval  $[\epsilon, 1 - \epsilon]$ . Note that  $\tilde{w}(1 - \epsilon, t) = 0$  is satisfied. This gives,

$$\begin{aligned} |z(t)|^p &= \left( \int_\epsilon^{1-\epsilon} w\varphi_1 \tilde{\rho} dr \right)^p \\ &\leq \|\varphi_1\|_\infty^p \left( \int_\epsilon^{1-\epsilon} |w| \tilde{\rho} dr \right)^p \\ &\leq C \left( |w(1 - \epsilon, t)| + \int_\epsilon^{1-\epsilon} |w'| \tilde{\rho} dr \right)^p \\ &\leq C |w(1 - \epsilon, t)|^p + C \left( \int_\epsilon^{1-\epsilon} |w'| \tilde{\rho} dr \right)^p \\ &= c_2 + CI_2, \end{aligned}$$

where  $c_2 := C|w(1 - \epsilon, t)|^p \rightarrow 0$  as  $\epsilon \rightarrow 0$ . We again used the uniform bound  $\|\varphi_1^\epsilon\|_\infty \leq C$ . Recalling (6.13), we obtain the differential inequality

$$\dot{z}(t) + \lambda_1 z(t) \geq \frac{1}{c_2}(|z(t)|^p - C) - c_1 \|u_0\|_\infty. \quad (6.14)$$

To obtain (6.9) from (6.14), it is sufficient that

$$\frac{1}{c_2} z(t)^p \geq \frac{1}{2} z(t)^p + \lambda_1^\delta z(t) + \frac{C}{c_2} + c_1 \|u_0\|_\infty, \quad (6.15)$$

where we have used that  $\lambda_1^\delta = \lambda_1((\delta, 1 - \delta)) \geq \lambda_1 = \lambda_1^\epsilon$  to avoid dependency on  $\epsilon$  for that coefficient. Furthermore, note that since  $z(t) \geq 0$  for all  $t \geq t_0$ , if  $z$  solves (6.9) in particular it is nondecreasing. Therefore, it is sufficient to verify (6.15) at  $t = t_0$ . To this end, fix  $1/c_2 \geq 2$ . The term on the left-hand side of (6.15) then dominates, so for a large enough value of  $z(t_0)$ , say

$$z(t_0) \geq M_1 > 0,$$

we have

$$\frac{1}{c_2} z(t_0)^p \geq \frac{1}{2} z(t_0)^p + \lambda_1 z(t_0) + \frac{1}{c_2} C. \quad (6.16)$$

We now set the largeness condition on  $u_0$ . Namely, that

$$\int_\delta^{1-\delta} \left( u_0(r) - \frac{1}{2} \|u_0\|_\infty \right) \hat{\varphi}(r) \hat{\rho}(r) dr \geq M := \max\{M_0, M_1, 1\} + 1 \quad (6.17)$$

Note that  $M = M(\Lambda, \lambda, p, n)$  only, so this condition may be set at the beginning of the proof, before choosing successively smaller  $\epsilon > 0$ . We handle the remaining term in (6.15) by setting

$$c_1 \leq \frac{1}{2} \int_\delta^{1-\delta} \hat{\varphi}(r) \hat{\rho}(r) dr. \quad (6.18)$$

This allows the term to be “absorbed” to the left later on. Note that this choice of  $c_1$  is possible because  $\hat{\varphi}, \hat{\rho}$  are both bounded by below by a positive constant in  $(\delta, 1 - \delta)$ , uniformly for small  $\epsilon$ . For this same reason, the largeness condition (6.17) is equivalent to the one given in the statement of Theorem 1.2.

Using the uniform convergence of the approximation  $w \rightarrow u$ , the uniform continuity of  $u$ , and that the integrand is nonnegative, we have

$$\begin{aligned} z(t_0) &= \int_\epsilon^{1-\epsilon} w(r, t_0) \hat{\varphi}(r) \hat{\rho}(r) dr \\ &\geq \int_\delta^{1-\delta} w(r, t_0) \hat{\varphi}(r) \hat{\rho}(r) dr \\ &\geq \int_\delta^{1-\delta} u_0(r) \hat{\varphi}(r) \hat{\rho}(r) dr - 1, \end{aligned}$$

hence  $z(t_0) \geq \max\{M_0, M_1, 1\}$ . Then,

$$\begin{aligned} z(t_0)^p - c_1 \|u_0\|_\infty &\geq z(t_0) - \frac{1}{2} \int_\delta^{1-\delta} \|u_0\|_\infty \hat{\varphi}(r) \hat{\rho}(r) dr \\ &\geq \int_\delta^{1-\delta} \left( u_0(r) - \frac{1}{2} \|u_0\|_\infty \right) \hat{\varphi}(r) \hat{\rho}(r) dr - 1 \\ &\geq \max\{M_0, M_1, 1\} > 0, \end{aligned}$$

Together with (6.16), this implies (6.15), hence  $z$  solves (6.9). Since  $z(t_0) \geq M_0$ , the conditions for blow-up are met. This gives the desired contradiction.  $\square$

*Remark 6.8.* The contradiction hypothesis, that the solution  $u$  satisfies the homogeneous boundary data in the classical sense, is very subtly used in the form of estimates depending on  $w(1 - \epsilon, t) \rightarrow 0$  for all  $0 \leq t \leq T$ .

*Remark 6.9.* We assume  $u_0 \in C^1(\overline{B_1(0)})$  since otherwise we do not know of the existence of solutions of the classical Dirichlet problem even for small  $T > 0$ , and the preceding argument fails.

We now use Theorem 1.2 to provide an example of LOBC for solutions of (1.1)-(1.3) in a more general bounded domain.

**Corollary 6.10.** *Let  $\Omega$  be a bounded domain satisfying a uniform interior sphere condition. Then, there exist  $u_0 \in C^1(\overline{\Omega})$ , with  $u_0 \geq 0$  and  $u_0|_{\partial\Omega} = 0$ , such that LOBC occurs for solutions of (1.1)-(1.3) in a finite time  $T = T(u_0, \Omega)$ .*

*Proof:* From the interior sphere condition, there exists an  $\eta > 0$  such that for all  $x_0 \in \partial\Omega$ , there exists a ball of radius  $\eta$  tangent to  $\partial\Omega$  at  $x_0$ , say  $B_\eta(x_1)$ . Consider  $\varphi \in C_0^\infty(B_1(0))$  a radial cut-off function such that

$$\varphi(r) = \begin{cases} 1, & r \leq \frac{2}{3} \\ 0, & r \geq \frac{3}{4} \end{cases}$$

and consider the solution

$$\begin{aligned} v_t - \mathcal{M}^-(D^2v) - |Dv|^p &= 0, & x \in B_1(0), & t > 0, \\ v &= 0, & x \in \partial B_1(0), & t > 0, \\ v(x, 0) &= C\varphi(|x|), & x \in \overline{B_1(0)}. \end{aligned}$$

It is easy to check that, for large enough  $C > 0$ ,  $C\varphi$  satisfies (1.16). Hence, by Theorem 1.2, LOBC occurs for  $v$  at some time  $T = T(C\varphi) > 0$ . As before,  $v$  is radial, thus  $v(x, T(C\varphi)) > 0$  for all  $x \in \partial B_1(0)$ .

We now rescale and translate  $v$  to obtain a solution in  $B_\eta(x_1)$ : define

$$\tilde{v}(x, t) = \eta^k v(\eta^{-1}|x - x_1|, \eta^{-2}t). \quad (6.19)$$

where  $k = \frac{p-2}{p-1}$ . Then  $\tilde{v}$  is a solution of (1.1) in  $B_\eta(x_1) \times \{t > 0\}$  with homogeneous boundary data and initial condition

$$\tilde{v}(x, 0) = C\eta^k \varphi(\eta^{-1}|x - x_1|). \quad (6.20)$$

The solution  $\tilde{v}$  is also radial with respect to  $x_1$ . Thus we can assume that  $\tilde{v}(x_0, T) > 0$ , where  $T = \eta^2 T(C\varphi)$ .

Define now

$$u_0(x) = \begin{cases} \tilde{v}(x, 0), & x \in B_\eta(x_1), \\ 0, & x \in \overline{\Omega} \setminus B_\eta(x_1), \end{cases}$$

and consider the solution  $u$  of

$$\begin{aligned} u_t - \mathcal{M}^-(D^2u) - |Du|^p &= 0, & x \in \Omega, \ t > 0, \\ u &= 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) &= u_0(x), & x \in \overline{\Omega}, \end{aligned}$$

with  $u_0$  as previously defined. Of course,  $u$  is also a solution of (1.1) in  $B_\eta(x_1) \times \{t > 0\}$ , and satisfies  $u \geq 0$  on  $\partial B_\eta(x_1) \times \{t > 0\}$ . Then by comparing the solutions in we have

$$u \geq \tilde{v} \text{ in } B_\eta(x_1) \times \{t > 0\}. \quad (6.21)$$

Hence,

$$u(x_0, T) \geq \tilde{v}(x_0, T) > 0,$$

i.e., LOBC occurs for  $u$ .  $\square$

The previous result might be rephrased to include a condition applicable to more general  $u_0$  than the example provided. We avoided this for simplicity, since the condition is rather convoluted, but do so now for completeness.

For any ball  $B_\eta(x_1) \subset \Omega$ , where  $x_1 \in \Omega$ ,  $\eta > 0$  and  $\partial B_\eta(x_1)$  is tangent to  $\partial\Omega$  at  $x_0$ , denote the radial variable by  $r = |x - x_1|$ . Note  $0 < r < \eta$ . For any  $v$  defined in  $B_\eta(x_1)$ , we may define the radial symmetrization

$$s(v)(r) = \inf_{\partial B_r(x_1)} v.$$

Note that, for any  $u_0 \in C(\overline{\Omega})$  such that  $u_0|_{\partial\Omega} = 0$ , we have  $s(u_0) \leq u_0$  in  $B_\eta(x_1)$  and  $s(u_0)(\eta) = s(u_0)(x_0) = 0$ . Note also that  $\|s(u_0)\|_\infty = \|u_0\|_\infty$ .

**Corollary 6.11.** *Using the notation above, as well as that of Theorem 1.2 and Corollary 6.10, there exists positive constants  $\delta = \delta(\lambda, \Lambda, n)$  and  $M = M(\lambda, \Lambda, n, p)$  such that LOBC occurs for all solutions of (1.1)-(1.3) with initial data  $u_0$  such that*

$$\sup \left\{ \eta^{-k} \int_\delta^{1-\delta} s(u_0)(\eta r) - \frac{1}{2} \|u_0\|_\infty dr \right\} > M. \quad (6.22)$$

Here, the supremum runs over all  $B_\eta(x_1) \subset \Omega$  tangent to  $\partial\Omega$  for fixed  $\eta > 0$ .

*Proof:* The corresponding proof is analogous to that of Corollary 6.10, in that it follows by a comparison argument and scaling between  $B_\eta(x_1)$  and  $B_1(0)$ . Hence, it will be omitted.  $\square$

*Remark 6.12.* After a change of variable, condition (6.22) can be written as

$$\sup \left\{ \int_{\eta\delta}^{\eta(1-\delta)} s(u_0)(r) - \frac{1}{2} \|u_0\|_\infty dr \right\} > \eta^{k+1} M,$$

which more closely resembles the condition (6.17) in Theorem 1.2, in that the limits of integration and the constant on the right-hand side reflect the dependence on  $\Omega$  (through  $\eta$ ), in addition to  $\lambda, \Lambda, n$  and  $p$ . Note that the supremum still runs over all interior tangent spheres  $B_\eta(x_1)$  for fixed  $\eta > 0$ .

## 7 Extensions

To extend the results regarding LOBC to more general equations, we must first guarantee that the global existence result of [7] applies to the equations considered. In fact, that the result applies to our model equation, (1.1), will follow as a particular case. For convenience, we restate part of what was mentioned in the introduction. Namely, consider

$$u_t - F(D^2u) = f(Du) \text{ in } \Omega \times (0, T),$$

where the nonlinearities are as follows:  $F : S(n) \rightarrow \mathbb{R}$  is uniformly elliptic, i.e.,

$$\mathcal{M}^-(X - Y) \leq F(X) - F(Y) \leq \mathcal{M}^+(X - Y) \quad \text{for all } X, Y \in S(n),$$

and vanishes at zero. In particular, this implies that

$$\mathcal{M}^-(X) \leq F(X) \leq \mathcal{M}^+(X) \quad \text{for all } X \in S(n). \quad (7.1)$$

The gradient nonlinearity  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies  $f(\xi) \geq |\xi|^2 h(|\xi|)$  for all  $\xi \in \mathbb{R}^n$ , where  $h : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$h = h(s) \text{ is positive nondecreasing for } s > 0, \quad (7.2)$$

$$s \mapsto s^2 h(s) \text{ is convex,} \quad (7.3)$$

$$h(yz) \leq C(h(y) + h(z)) \text{ for large } y, z > 0 \text{ and some } C > 0. \quad (7.4)$$

The last condition implies that  $h$  grows more slowly than any positive power. Examples which satisfy the conditions above are  $h(s) = (\log s)^p$  and  $h(s) = (\log s)^p (\log \log s)^q$ , for large  $s$  and  $p, q > 0$ .

### 7.1 Comparison, existence and uniqueness

We look to verify hypotheses (H1) and (H2) needed for the Strong Comparison Principle, Theorem 2.1. We begin by setting

$$G(x, r, \xi, X) = G(\xi, X) = -F(X) + f(\xi), \quad (7.5)$$

where  $\xi \in \mathbb{R}^n$ ,  $X \in S(n)$ , and the nonlinearities  $F, f$  (and consequently,  $h$ ) are as above. Note that this is compatible with the exchange of sub- and supersolutions mentioned in Remark 2.3, since  $\tilde{F} : S(n) \rightarrow \mathbb{R}$  given by

$$\tilde{F}(X) = -F(-X), \quad \text{for all } X \in S(n)$$

is uniformly elliptic and vanishes at zero if  $F$  does.

We proceed to check (i) through (iii) of property (P) for  $h_1(s) = s^2 h(s)$ , with  $h$  as above:

(i) By the growth condition on  $h$ , we have

$$\int^\infty \frac{s}{h_1(s)} ds = \int^\infty \frac{s}{s^2 h(s)} ds = \int^\infty \frac{1}{s h(s)} ds < \infty.$$

(ii) Since  $h$  is nondecreasing,

$$L \mapsto L^2 s^2 h(Ls) - CL^2 s^2 h(s) = L^2 s^2 (h(Ls) - Ch(s))$$

is increasing for all  $s > 0$  and  $L \geq 1$ .

(iii) Since  $L, s > 0$  will be taken large, it is equivalent to show that, for fixed  $C, \tilde{C} > 0$  and  $\epsilon > 0$ ,

$$\begin{aligned} h(Ls) - Ch(s) &\geq \epsilon, \\ \epsilon &> \tilde{C}/Ls. \end{aligned}$$

It follows from the growth condition on  $h$  that  $h(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . Hence, fixing  $s > 0$  and taking large enough  $L = L(s) > 1$ , we get  $h(Ls) \geq \epsilon + Ch(s)$ . The second inequality above comes from choosing  $L$  large as well.

This shows (1.17) satisfies (H1). Now on to (H2). The second matrix inequality (2.6) in (H2) implies  $X \leq Y + o(1)$ . Hence,  $\mu X \leq Y + o(1)$  for any  $0 < \mu < 1$ , and from the uniform ellipticity and the definition of the Pucci operator, this gives

$$F(\mu X) - F(Y) \leq \mathcal{M}^+(\mu X - Y) \leq o(1).$$

For the contribution of the gradient term to the estimate of (H2), it suffices to have  $h_1$  above (i.e.,  $s \mapsto s^2 h(s)$ ) be locally Lipschitz and satisfy the following, as noted in Example 1 of [7]:

(H3) For all  $C > 0$ , there exists a sequence  $0 < \mu_\epsilon < 1$  defined for  $0 < \epsilon \leq 1$  such that  $\mu_\epsilon \rightarrow 1$  as  $\epsilon \rightarrow 0$  and such that for all large  $r > 0$  large enough and  $0 < \epsilon$  small enough, we have:

$$C\epsilon r \sup_{0 \leq \tau \leq r(1+C\epsilon)} |h'_1(\tau)| \leq (1 - \mu_\epsilon) \inf_{\tau \geq r(1-C(1-\mu_\epsilon))} (h'_1(\tau)\tau - h_1(\tau)).$$

Given the properties of  $h$  above, a lengthy but straightforward computation shows that to verify (H3) it suffices to choose  $\mu_\epsilon$  such that  $\epsilon^{-1}(1 - \mu_\epsilon) \rightarrow +\infty$  as  $\epsilon \rightarrow 0$ .

## 7.2 Loss of boundary conditions

What follows is an extension of Theorem 1.2 that includes a more general gradient term, with a suitable growth condition. Consider

$$u_t - \mathcal{M}^-(D^2u) = g(|Du|) \text{ in } B_1(0) \times (0, T), \quad (7.6)$$

where  $g : \mathbb{R} \mapsto \mathbb{R}$ ,  $g$  is convex increasing for  $s \geq 0$ , and  $g(0) = 0$ . Note that (7.6) has no “ $(x, t)$ -dependence”, so that Lemma 4.4 applies directly. Lemma 5.1 applies as well if  $u_0$  is radially symmetric, given that  $g = g(|Du|)$ .

On the other hand, Lemma 4.5 requires a slight adaptation. In the case that  $w(\hat{x}, \hat{t}) < u_{\epsilon, \kappa}(\hat{x}, \hat{t})$ , where  $(\hat{x}, \hat{t})$  is a point of second-order differentiability of the regularized function  $w$ , we must take  $0 < \delta = \delta(g)$  small enough so that

$$\begin{aligned} w_t(\hat{x}, \hat{t}) - \mathcal{M}^-(D^2w(\hat{x}, \hat{t})) - g(|Dw(\hat{x}, \hat{t})|) &\geq -\frac{K}{\kappa^{\frac{1}{2}}} + \frac{\lambda}{\delta} - \frac{\Lambda(n-1)}{\epsilon} - g\left(\frac{K}{\epsilon^{\frac{1}{2}}}\right) \\ &\geq 0 \end{aligned}$$

to get that  $w$  is a supersolution.

For the statement of the Theorem, we define

$$a(s) = \sup_{y>0} \frac{g^{-1}(ys)}{g^{-1}(y)}, \quad s \geq 0,$$

and recall the definition of the convex conjugate,

$$g^*(s) = \sup \{ ys - g(y) \mid y \in \mathbb{R} \}.$$

**Lemma 7.1.** *Let  $u_0 \in C(\overline{B_1(0)})$  be a radial function, and  $g$  as described above. Assume also that  $g$  is such that (7.6) satisfies (H1) and (H2) of Section 2. If*

$$\int_1^\infty \frac{g^*(La(s))}{s^2} ds < \infty \quad (7.7)$$

*for all  $L > 0$ , then there exist positive constants  $\delta$  and  $M$ , depending only on  $\lambda, \Lambda, n$  and  $g$ , such that, if*

$$\int_\delta^{1-\delta} u_0(r) - \frac{1}{2} \|u_0\|_\infty dr > M, \quad (7.8)$$

*then the solution  $u$  of (7.6), (1.2), (1.3) with  $\Omega = B_1(0)$  and initial data  $u_0$  has LOBC at some finite time  $T = T(u_0)$ .*

*Proof:* Aside from using all the auxiliary results leading to Theorem 1.2, the proof follows that of Theorem 5.2 in [29]. We repeat most of the argument for convenience. Once more, we proceed by contradiction, assuming  $u$  is a solution which satisfies (1.2) in the classical sense. We consider  $\varphi_1$  as previously defined, and again denote by  $w$  the function obtained by regularizing the radial part of

the solution  $u$  of (7.6) for  $\Omega = B_1(0)$ . This function now satisfies, for arbitrary  $\epsilon > 0$  and  $0 < t_0 \leq t_1 < T$ ,

$$\tilde{\rho}w_t \geq (\rho w')' + \tilde{\rho}g(|w'|) \quad \text{for a.e. } r \in (\epsilon, 1 - \epsilon), t \in (t_0, t_1).$$

From the definition of  $a$ , setting  $y = g(|w'(r, t)|)\varphi_1(r, t)$  for  $r \in (\epsilon, 1 - \epsilon)$ ,  $t_0 < t < t_1$ , we have

$$a(1/\varphi_1)g^{-1}(g(|w'|)\varphi_1) \geq g^{-1}(y/\varphi_1) = |w'|.$$

Hence,

$$g(|w'|)\varphi_1 \geq g\left(\frac{|w'|}{a(1/\varphi_1)}\right). \quad (7.9)$$

Let  $L > 0$  to be chosen later. By the definition of the convex conjugate, we have

$$L|w'| = \frac{|w'|}{a(1/\varphi_1)}La(1/\varphi_1) \leq g\left(\frac{|w'|}{a(1/\varphi_1)}\right) + g^*(La(1/\varphi_1))$$

for a.e.  $r \in (\epsilon, 1 - \epsilon)$ ,  $t_0 < t < t_1$  (we have omitted the arguments for simplicity). This is an instance of Fenchel's inequality, and is the analogue of the Hölder estimate which appears in the proof of Theorem 1.2.

Using (7.9), we have

$$L \int_{\epsilon}^{1-\epsilon} |w'| \tilde{\rho} dr \leq \int_{\epsilon}^{1-\epsilon} g(|w'|)\varphi_1 \tilde{\rho} dr + \int_{\epsilon}^{1-\epsilon} g^*(La(1/\varphi_1)) \tilde{\rho} dr.$$

That the second integral is finite follows from (7.7). It can also be proven that it is bounded uniformly in  $\epsilon$  as in Lemma 6.3.

Arguing as in the beginning of the proof of Theorem 1.2, we obtain

$$\dot{z}(t) + \lambda_1 z(t) \geq I - c_1 \|u_0\|_{\infty} - C,$$

where  $z$  is defined exactly as before, but now

$$\begin{aligned} I &:= \int_{\epsilon}^{1-\epsilon} g(|w'|)\varphi_1 \tilde{\rho} dr - C \\ &\geq L \int_{\epsilon}^{1-\epsilon} |w'| \tilde{\rho} dr - C \\ &\geq L \left( \int_{\epsilon}^{1-\epsilon} |w| \tilde{\rho} dr - w(1 - \epsilon, t) \right) - C. \end{aligned}$$

Here,  $L > 0$  is yet to be chosen;  $c_1$ ,  $w(1 - \epsilon, t) \rightarrow 0$  as  $\epsilon \rightarrow 0$ ; and  $C$  denotes a generic constant that does not depend on  $\epsilon, t_0, t_1$ , nor  $\|u_0\|_{\infty}$ , and might change from line to line as long as this uniformity remains.

Appropriately choosing constants, we reduce the previous ODE to

$$\dot{z}(t) \geq z(t) - C \quad \text{for a.e. } t \in (t_0, t_1), \quad (7.10)$$



which we can integrate to get

$$z(t) \geq (z(t_0) - C)e^{t-t_0} \quad \text{for all } t \in (t_0, t_1), \quad (7.11)$$

To conclude, there is a slight change in our strategy, owing to the difference in the ODE satisfied by  $z = z(t)$ . Note that (7.10) does not blowup in finite time, as does (6.9). We will find a contradiction in the form of a bound, derived from (7.11), which we can easily violate by choosing the appropriate  $u_0$ . Also, since our aim is to prove nonexistence beyond some finite time, we may assume  $T > 0$  is large to achieve this contradiction. By choosing the time-regularization parameter small as well, the difference  $t_1 - t_0$  is large as well, say  $t_1 - t_0 \geq T/2$ .

Recall the bounds

$$\hat{\varphi}(r) \leq C \quad \text{and} \quad \hat{\rho}(r) \leq \frac{1}{\lambda}, \quad \text{for all } r \in (0, 1),$$

which come from the compactness lemma (Lemma 3.1 in [14]) and (5.5), respectively. We assume  $T > 0$  large enough so that

$$e^{-\frac{T}{2}} < \min \left\{ \frac{\lambda}{2C} \int_{\delta}^{1-\delta} \hat{\varphi}(r) \hat{\rho}(r) dr, \frac{\lambda}{C} \right\} \quad (7.12)$$

Note that this  $T$  depends only on  $\lambda, \Lambda$  and  $n$ .

As before, using the uniform convergence  $w \rightarrow u$ , the uniform continuity of  $u$ , and that the integrand is nonnegative, we have

$$z(t_0) \geq \int_{\delta}^{1-\delta} u_0(r) \hat{\varphi}(r) \hat{\rho}(r) dr - 1.$$

On the other hand, from evaluating (7.11) at  $t = t_1$ ,

$$\begin{aligned} z(t_0) &\leq e^{-(t_1-t_0)} z(t) + C \\ &\leq e^{-\frac{T}{2}} \int_{\epsilon}^{1-\epsilon} w(r, t) \hat{\varphi}(r) \hat{\rho}(r) dr + C \\ &\leq e^{-\frac{T}{2}} \frac{C}{\lambda} (\|u_0\|_{\infty} + 1) + C \\ &\leq \int_{\delta}^{1-\delta} \frac{1}{2} \|u_0\|_{\infty} \hat{\varphi}(r) \hat{\rho}(r) dr + C. \end{aligned}$$

The second inequality corresponds to our choice of  $t_0, t_1$  and the definition of  $z$ , the third uses uniform convergence and the bounds for  $\hat{\varphi}$  and  $\hat{\rho}$ , while the fourth inequality uses (7.12). The value of  $C$  changes in the last line, but still depends on the appropriate quantities. In particular it does not depend on  $T$  (this is the reason for using the second part of the min in (7.12)).

Combining both estimates for  $z(t_0)$ , we have

$$\int_{\delta}^{1-\delta} (u_0 - \frac{1}{2} \|u_0\|_{\infty}) \hat{\varphi}(r) \hat{\rho}(r) dr < C,$$

where  $C = C(\lambda, \Lambda, n, g)$ . This bound is readily violated by choosing a suitably large  $u_0$ , and as before, it is equivalent to the one in the statement of the lemma since  $\hat{\varphi}$  and  $\hat{\rho}$  are uniformly bounded from below.  $\square$

Finally, we proceed with the proof of the extension mentioned in the introduction.

*Proof of Theorem 1.3:* First, define  $g(s) = s^2 h(s)$ , where  $h$  satisfies (7.2)-(7.4) and the growth condition (1.19). It is proven in Lemma 5.3 and the Completion of Theorem 2.2 in [29] that  $g$  so defined satisfies the hypothesis of Theorem 7.1, including (7.7). Furthermore, using (7.1), if  $u$  is a solution of (1.17), formally we have that

$$\begin{aligned} u_t - \mathcal{M}^-(D^2 u) &\geq u_t - F(D^2 u) = f(Du) \\ &\geq |Du|^2 h(|Du|) = g(|Du|) \text{ in } \Omega \times (0, T), \end{aligned}$$

and this is readily checked using test functions. Hence,  $u$  is a supersolution of (7.6) in  $\Omega \times (0, T)$ . Using the interior sphere condition, without loss of generality we may assume that  $B_1(0) \subset \Omega$  with  $B_1(0)$  tangent to  $\partial\Omega$  at some point  $x_0 \in \partial\Omega \cap \partial B_1(0)$  (This is equivalent to repeating the constructions of Theorems 1.2 and 7.1 on a ball of arbitrary radius and performing translation.) We then conclude that  $u$  is a supersolution of (7.6) in  $B_1(0) \times (0, T)$ . Let  $\tilde{u}_0 \in C(\overline{B_1(0)})$  nonnegative, radially symmetric and satisfies (7.8), and consider the solution  $\tilde{u}$  of (7.6) with initial data  $\tilde{u}_0$  and homogeneous boundary data. By Lemma 7.1,  $\tilde{u}$  has LOBC in finite time  $T' > 0$ , and since it is radially symmetric we may conclude that LOBC occurs at  $x_0 \in B_1(0)$ . Now define

$$u_0(x) = \begin{cases} \tilde{u}_0(x), & x \in B_1(0) \\ 0, & x \in \overline{\Omega} \setminus B_1(0). \end{cases}$$

By comparison, we have that the solution  $u$  of (1.17)-(1.2)-(1.3) with initial data  $u_0$  satisfies  $u(x_0, T') \geq \tilde{u}(x_0, T') > 0$ , hence  $u$  has LOBC.  $\square$

*Remark 7.2.* The more general nonlinearities do not admit a rescaling argument like the one given in the proof of Corollary 6.10. Furthermore, (1.17) may no longer have radial symmetry. The preceding argument is in some sense simpler, but it does not provide a condition one can check for any given initial data like the one in Corollary 6.11.

*Remark 7.3.* The typical example for the nonlinearity  $h$  in Theorem 1.3 is  $h(s) = (\log s)^p$  for  $p > 0$  and large  $s$ . In this case, the growth condition (1.19) forces that  $p > 1$ . This is consistent with what is known to be a more precise condition for preventing GBU in the case of the viscous Hamilton-Jacobi equation: for

$$u_t - \Delta u = f(u, \nabla u) \quad \text{in } \Omega \times (0, T),$$

GBU does not occur if

$$|f(u, \nabla u)| \leq C(u)(1 + |\nabla u|^2)h(|\nabla u|)$$

where  $C(u)$  is locally bounded, and  $h$  is positive nondecreasing and satisfies

$$\int_1^\infty \frac{1}{sh(s)} = \infty.$$

See [27], Ch. IV, and the references therein.

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